# Hidden Substitutes*

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**Abstract**

We show that, in the setting of many-to-one matching with contracts, preferences exhibiting some forms of complementarity have an underlying substitutable structure. In particular, we identify "hidden" substitutabilities in agents' preferences; this makes stable and strategy-proof matching possible in new settings with complementarities, even though stable outcomes are not guaranteed, in general, when complementarities are present. Our results give new insight into a range of real-world market design settings.

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# 1 Introduction

Stability and strategy-proofness are key goals of practical market design: Stability, like the core, rules out the possibility that agents may profitably recontract after market-clearing; this reduces both unraveling and costly *ex post* renegotiation (Roth, 1984, 1990; Roth and Xing, 1994; Kagel and Roth, 2000; Avery et al., 2001, 2007) while guaranteeing a form of fairness (Balinski and Sönmez, 1999; Abdulkadiroğlu and Sönmez, 2003). Meanwhile, strategy-proofness ensures that truthful reporting is a dominant strategy; this eliminates the gains to strategic manipulation, both simplifying participation and ensuring that allocations are calculated using accurate preference data (Pathak and Sönmez, 2008; Pathak, 2017).[1] Guarantees about stability and strategy-proofness must be made upfront, as market mechanisms have to be established prior to preference elicitation. Thus, much of the research in matching market design has focused on characterizing when stable and strategy-proof matching can be guaranteed.

In many settings, ranging from matching to auctions to exchange economies with discrete goods, achieving stable outcomes (or, equivalently, competitive equilibria) requires ruling out complementarities across contractual offers—that is, it is necessary that each agent view offers as *substitutes* for each other, in the sense that a new offer will never lead an agent to demand an offer he would otherwise reject.[2] But surprisingly, there is one setting in which substitutability is *not* necessary for stable and strategy-proof matching: *many-to-one matching with contracts*, in which workers match to firms while negotiating contractual terms, with each firm potentially employing multiple workers (but with each worker taking at most

[1] Strategy-proof mechanisms elicit—and thus base assignment upon—true preferences in equilibrium. Additionally, by eliminating the gains from strategic sophistication, strategy-proofness ensures "equal access" to the mechanism (Pathak and Sönmez, 2008).

[2] Substitutable preferences are necessary to guarantee the existence of stable outcomes in the settings of many-to-one matching without contracts (Hatfield and Kojima, 2008), many-to-one matching with transfers (Gul and Stacchetti, 1999; Hatfield and Kojima, 2008), many-to-many matching with and without contracts (Hatfield and Kominers, 2017), matching in vertical networks (Hatfield and Kominers, 2012), and matching in trading networks with transfers (Hatfield et al., 2013). Substitutable preferences are also key in auction settings (Milgrom, 2004, 2007; Ausubel and Milgrom, 2006; Klemperer, 2010) and exchange economies (Bikhchandani and Mamer, 1997; Gul and Stacchetti, 1999, 2000; Sun and Yang, 2006).

one contract). Substitutability was originally believed to be necessary for guaranteeing the existence of stable outcomes in many-to-one matching with contracts—indeed, the Hatfield and Milgrom (2005) paper that presented the many-to-one matching with contracts model was published with an incorrect claim (p. 921) to that effect.[3]

The fact that substitutability is not necessary for stable many-to-one matching with contracts has turned out to be vital in understanding many real-world applications, including cadet–branch matching (Sönmez and Switzer, 2013; Sönmez, 2013), the design of affirmative action mechanisms (Kominers and Sönmez, 2015), airline upgrade allocation (Kominers and Sönmez, 2015), and the matching of lawyers to traineeships in Germany (Dimakopoulos and Heller, 2019). Yet the reasons that stable many-to-one matching with contracts does not require substitutability have remained opaque.

This paper explains a large part of the mystery, and in doing so unlocks a host of new applications. We show that a large class of non-substitutable preferences have a "hidden" underlying substitutable structure—effectively, they can be understood as projections of substitutable preferences from a more complex setting. All of the applications discussed in the preceding paragraph have this hidden substitutable structure.[4] Moreover, a number of new applications rely explicitly on the methods and insights introduced here: Hassidim et al. (2016a,b, 2017) have used our results in redesigning the Israeli Psychology Masters Match. Meanwhile, Aygün and Turhan (2016, 2017) have used our work to propose a new procedure to allocate students across the Indian Institutes of Technology, and Yenmez (2018) used our approach to develop a new mechanism for centralized university admissions. Hassidim et al. (2018) have also recently used our framework to incorporate budget constraints into stable matching in a straightforward manner; this result is particularly surprising given that budget constraints are typically quite difficult to integrate into market design frameworks.[5]

[3] Hatfield and Kojima (2008) were the first to notice the error of Hatfield and Milgrom (2005).

[4] In particular, all of those applications fit within the slot-specific priorities framework of Kominers and Sönmez (2015), which our framework encapsulates (see Appendix E and Appendix E).

[5] In auction settings, for example, budget constraints introduce a great deal of complexity (Che and Gale, 1998; Benoît and Krishna, 2001; Milgrom, 2004; Pai and Vohra, 2014).

Hatfield and Kojima (2010) were the first to identify a class of non-substitutable preferences for which stable and strategy-proof many-to-one matching with contracts is possible; this work proved useful in applications such as cadet–branch matching (mentioned above; see Sönmez and Switzer (2013) and Sönmez (2013)). Substitutable completability subsumes Hatfield and Kojima's (2010) condition for stable and strategy-proof matching (see Section 5.7 and Kadam (2017)).[6] Moreover, substitutable completability is both more intuitive than prior weakened substitutability concepts and enables a broad new class of applications (see Section 5).

The remainder of this paper is organized as follows: Section 2 presents an elementary example illustrating how substitutable completability leads to stable outcomes. Section 3 then presents the general model of many-to-one matching with contracts. Section 4 formally defines substitutable completability and presents our main results. Section 5 discusses market design applications of substitutable completability that we and others have developed. Section 6 concludes. Proofs and other supplementary materials are presented in the Appendix.

# 2 An Elementary Example

We illustrate the main idea of the paper with what we might call an "elementary" example: We consider a setting with two doctors, Sherlock ($s$) and Watson ($w$), and one hospital $h$ (conveniently located near Baker Street). Sherlock is a genius who can take any job in a hospital—he can do either research work ($r$) or clinical work ($c$). Watson, meanwhile, can only do clinical tasks, and cannot do them as well as Sherlock can. Hospital $h$ would like to have both a researcher and a clinician, but prefers to have a clinician if only one doctor is available.

A contract between a doctor $d$ and the hospital $h$ is a triple $(d, h, t)$, which specifies the type of employment $t \in \{r, c\}$. Each doctor can take at most one contract. We can represent

[6] Figure 1 in Appendix D shows the relationship between our work and prior weakened substitutability concepts introduced in the literature.

the preferences of the hospital $h$ over sets of contracts with the following preference relation:

$$\{(s,h,r),(w,h,c)\} \succ_h \{(s,h,c)\} \succ_h \{(w,h,c)\} \succ_h \{(s,h,r)\} \succ_h \varnothing, \tag{1}$$

where we interpret $Y \succ_h Z$ as meaning that hospital $h$ prefers the set of contracts $Y$ to the set of contracts $Z$ (and where an unlisted set $W$ is unacceptable, in the sense that $h$ prefers $\varnothing$ to $W$).

Note that contracts $(s,h,r)$ and $(w,h,c)$ are *complements* for $h$: When hospital $h$ has both contracts with Sherlock available ($(s,h,r)$ and $(s,h,c)$) but no opportunity to hire Watson, the hospital $h$ will take the contract $(s,h,c)$ instead of $(s,h,r)$. Yet when $(w,h,c)$ becomes available, hospital $h$ prefers to take $(s,h,r)$ instead of $(s,h,c)$, while simultaneously also taking $(w,h,c)$. Thus, $(s,h,r)$ is complementary with $(w,h,c)$ in the sense that $(s,h,r)$ is taken instead of $(s,h,c)$ once $(w,h,c)$ becomes available—hospital $h$ prefers to hire Sherlock as a researcher instead of as a clinician only if it can hire Watson as a clinician instead.

Complementarities like those arising in $h$'s preferences can be problematic because they often preclude the existence of a natural equilibrium outcome. Indeed, we say that an outcome is *stable* if it is *individually rational*, i.e., no agent wishes to unilaterally drop a contract, and *unblocked*, i.e., no set of agents can profitably recontract among themselves.[7] Stability has been shown to be practically important for the success of real-world medical residency assignment programs (Roth and Xing, 1997; Roth, 2002), as well as school choice systems (Abdulkadiroğlu, Pathak, Roth, and Sönmez, 2005; Abdulkadiroğlu, Pathak, and Roth, 2005). But when hospitals' preferences involve complementarities, stable outcomes generally do not exist.

When a hospital's preferences do not exhibit complementarities, we say that they are *substitutable*. Most matching models assume substitutability in order to guarantee the existence of stable outcomes. Yet the preferences for $h$ described in (1) seem quite natural, and we would like to be able to incorporate them into a stable matching framework. Luckily, just as in a good mystery novel, not everything in our Sherlock–Watson example is as it

[7]In our setting, stability is in fact equivalent to the core; see Appendix B.2.

seems.

It turns out that regardless of Sherlock's and Watson's preferences over contracts, a stable outcome always exists when $h$'s preferences are given by (1). For instance, suppose that Sherlock prefers clinical work to research (and finds both contracts acceptable), i.e.,

$$\{(s,h,c)\} \succ_s \{(s,h,r)\} \succ_s \varnothing,$$

while Watson prefers to be employed rather than not, i.e.,

$$\{(w,h,c)\} \succ_w \varnothing.$$

In this case, there are two stable outcomes:

- In the first stable outcome, Sherlock is employed as a clinician and Watson is unemployed (i.e., $\{(s,h,c)\}$ is stable). In particular, $\{(s,h,c)\}$ is unblocked, as Sherlock prefers the clinician contract to the researcher contract, and the hospital does not desire to hire Watson if it employs Sherlock as a clinician.

- In the second stable outcome, Sherlock is employed as a researcher and Watson is employed as a clinician (i.e., $\{(s,h,r),(w,h,c)\}$ is stable). In particular, $\{(s,h,r),(w,h,c)\}$ is unblocked as with that set the hospital obtains its favorite set of contracts.[8]

More generally, it is possible to check by hand that the set of stable outcomes is non-empty for any specification of preferences for Sherlock and Watson (although finding the stable outcomes in each case requires a bit of brute-force detective work). What is going on is that the complementarity in hospital $h$'s preferences is in some sense illusory—it is hiding a deeper substitutable structure.

A more complete view of $h$'s preferences recognizes that because Sherlock is the best at each task, hospital $h$, if it could, *would most prefer to hire two Sherlocks—one as a researcher,*

[8]The other six possible outcomes are not stable: Both $\{(s,h,r),(s,h,c)\}$ and $\{(s,h,r),(s,h,c),(w,h,c)\}$ are not individually rational for Sherlock. The outcome $\{(s,h,c),(w,h,c)\}$ is not individually rational for the hospital $h$. Finally, the outcomes $\{(s,h,r)\}$, $\{(w,h,c)\}$, and $\varnothing$ are all blocked by $\{(s,h,c)\}$.

*and one as a clinician*. Thus, we can "*complete*" $h$'s preference relation by extending it to the following "extended" preference relation $\hat{\succ}_h$, which imagines that Sherlock can take both jobs:

$$\boldsymbol{\{(s,h,r),(s,h,c)\}} \mathrel{\hat{\succ}_h} \{(s,h,r),(w,h,c)\} \mathrel{\hat{\succ}_h} \{(s,h,c)\} \mathrel{\hat{\succ}_h} \{(w,h,c)\} \mathrel{\hat{\succ}_h} \{(s,h,r)\} \mathrel{\hat{\succ}_h} \varnothing. \tag{2}$$

Note that (2) differs from (1) only in the first, bolded component—and the bolded set $\{(s,h,r),(s,h,c)\}$ is not individually rational for Sherlock, as it requires him to take two contracts whereas he desires at most one.

As a consequence, we infer that any outcome stable under $h$'s extended preferences (2) must also be stable under $h$'s true preferences (1).[9] To see this, note that if an outcome $A$ is stable under $h$'s extended preferences (2), then $A$ must still be individually rational for the doctors, and so $A$ must contain at most one contract with each doctor—even though a set can be individually rational for $h$ under (2) while containing two contracts with Sherlock. Thus, since $A$ must contain at most one contract with each doctor, the individual rationality of $A$ under (2) implies that $A$ is also individually rational under (1). Finally, if $A$ were blocked under $h$'s true preferences (1), then $A$ would also be blocked under $h$'s extended preferences (2).

It is also possible to verify that under the extended preference relation (2), the hospital $h$ regards all contracts as substitutes. Recall that $h$'s preferences under (1) were not substitutable as $h$ took $(s,h,c)$ instead of $(s,h,r)$ when only those two contracts were available, but took $(s,h,r)$ once $(w,h,c)$ became available as well. But the complementarity between $(w,h,c)$ and $(s,h,r)$ disappears under (2) because the hospital always takes $(s,h,c)$ and $(s,h,r)$ when both contracts are available. Thus, we have uncovered "hidden" substitutable structure in $h$'s preferences.

It is by now well-known that stable outcomes are guaranteed to exist under substitutable preferences (Kelso and Crawford, 1982; Hatfield and Milgrom, 2005; Ostrovsky, 2008;

[9] In fact, as we show in our Lemma 1, an outcome is stable under $h$'s true preferences if and only if it is stable under $h$'s extended preferences.

Hatfield and Kominers, 2012, 2017). We combine this fact with the previous observation that any outcome stable under the completed preferences (2) is stable under the original preferences (1) to show that stable outcomes are guaranteed to exist under $h$'s original preferences (1).

Additionally, the availability of a substitutable completion like (2) gives us a way to find stable outcomes using a variant of the celebrated deferred acceptance algorithm of Gale and Shapley (1962), which produces stable outcomes for any substitutable preference structure. Moreover, under an additional regularity condition (the Law of Aggregate Demand, which is satisfied in our Sherlock–Watson example), stable matching can be made strategy-proof for doctors, in the sense that it is a weakly dominant strategy for doctors to report their preferences truthfully—generalizing results of Dubins and Freedman (1981) and Hatfield and Milgrom (2005).

The remainder of the paper generalizes the insights presented here, showing that when preferences of hospitals have hidden substitutable structure, we can extend the key methods and results of matching theory. Moreover, as we describe in our application sections, hidden substitutability arises in a number of real-world applications where stable and strategy-proof matching is crucial.

# 3 Model

We work with the Hatfield and Milgrom (2005) *many-to-one matching with contracts* model, in which doctors and hospitals match to each other while negotiating contractual terms. There is a finite set $D$ of *doctors*, a finite set $H$ of *hospitals*, and a finite set $T$ of *contractual relationships*.[10] A *contract* $x = (d, h, t)$ is a triple specifying a doctor $d$, a hospital $h$, and a contractual relationship $t$. The set of all possible contracts, which we denote $X$, is then a subset of $D \times H \times T$.

For any set of contracts $Y \subseteq X$ and any doctor $d \in D$, we let $Y_d$ denote the set of

[10] In practice, a contractual relationship can encode terms such as wages, work hours, and responsibilities.

contracts associated with $d$. Similarly, for any set of contracts $Y \subseteq X$ and any $h \in H$, we let $Y_h$ denote the set of contracts associated with $h$. We assume that doctors have *unit demand*; thus we say that a set of contracts is *feasible* if $|Y_d| \leq 1$ for all doctors $d \in D$.

Each agent $i$ is endowed with a *(many-to-one) preference relation* $\succ_i$, which is a strict linear order over feasible sets of contracts involving that agent; these preferences naturally carry over to all feasible subsets of $X$, where, for any two feasible sets of contracts $Y$ and $Z$, we say that $Y \succeq_i Z$ if $Y_i \succeq_i Z_i$.[11] A set of contracts $Y$ is *acceptable to* $i$ if $Y_i \succeq_i \varnothing$. We say that the set $A$ is the *most-preferred set from* $Y$ *under* $\succ_i$ if $A$ is acceptable, $A \subseteq Y_i$ and, for every $A' \subseteq Y$, we have that $A \succeq_i A'$.[12] In other words, the most-preferred set from $Y$ under $\succ_i$ is simply the highest-ranked set of contracts according to $\succ_i$ that is composed only of contracts in $Y$ associated with $i$.

## 3.1 Outcomes

In our framework, an *outcome* is just a set of contractual obligations for each agent; hence, an outcome can be specified by a set of contracts $A \subseteq X$. The central equilibrium concept of matching theory is *stability*, which imposes two conditions on outcomes:

1. A stable outcome $A$ must be *individually rational*, in the sense that no agent $i$ wishes to unilaterally abrogate any of his contracts in $A$; formally, $A$ is *individually rational for* $i$ under $\succ_i$ if $A_i$ is $i$'s most-preferred set from $A_i$.

2. A set of contracts $Z$ associated with a single hospital $h$ *blocks* $A$ if $h$ (strictly) prefers $Z$ to $A$ and each doctor with a contract in $Z$ (weakly) prefers $Z$ to $A$; formally, $Z \subseteq X_h$ *blocks* $A$ if $Z \succ_h A$ and $Z_d \succeq_d A_d$ for all doctors $d$ such that $Z_d \neq \varnothing$. A stable outcome $A$ must be *unblocked*, in the sense that there are no blocks for $A$.

As we show in Appendix B.2, the stability concept we use here is equivalent to the core

[11] In Appendix A, we generalize all of the concepts and results of Sections 3 and 4 to a setting with choice functions that will be more familiar to readers specialized in matching theory.

[12] Note that for any set of contracts $Y \subseteq X$, a most-preferred set from $Y$ exists for any $\succ_i$ as $\succ_i$ is a strict linear order and the empty set is both a subset of every $Y$ and is always acceptable.

defined by weak domination (sometimes called the strict core).[13]

Stability is important for a number practical reasons: Stability eliminates the incentives for *ex post* recontracting (Roth, 2002, 2009; Kominers et al., 2017). Moreover, stable matching mechanisms have been found to reduce *unraveling*, under which offers are made earlier and earlier, leading to inefficient contracting in markets such as medical residency and law clerk hiring (Roth, 1990; Roth and Xing, 1994; Kagel and Roth, 2000; Avery et al., 2001, 2007).

## 3.2 Conditions on Preference Relations

Much of matching theory depends heavily on the assumption that contracts are substitutes, in the sense that gaining a new offer $x$ can not make an agent $i$ choose a contract $z$ that $i$ would otherwise reject. In other words, substitutability requires that no two contracts $x$ and $z$ are "complements," in the sense that access to $x$ makes $z$ desirable, whereas without access to $x$, the contract $z$ is undesirable. Formally, a preference relation $\succ_i$ is *substitutable* if, for all sets of contracts $Y$ and all distinct contracts $x$ and $z$, if $z$ is not in the most-preferred set from $Y \cup \{z\}$ under $\succ_i$, then $z$ is not in the most-preferred set from $\{x\} \cup Y \cup \{z\}$ under $\succ_i$. In our framework, doctors' preferences are always substitutable since doctors have unit demand.

The Law of Aggregate Demand, first introduced by Hatfield and Milgrom (2005), is a monotonicity condition that requires an agent chooses a larger set of contracts when more contracts become available.[14] Formally, a preference relation $\succ_i$ satisfies the *Law of Aggregate Demand* if, the most-preferred set from $Y$ is weakly larger than the most-preferred set from $\hat{Y}$ whenever $Y \supseteq \hat{Y}$. In our framework, doctors' preferences always satisfy the Law of Aggregate Demand, as doctors have unit demand.

[13] The concept of blocking we use here is based on the core rather than the standard concept of blocking used throughout matching theory, which generalizes more directly to settings with choice functions. As we show in Appendix B.2, the definition we use here is equivalent to the standard stability concept for many-to-one matching with contracts, which says that $A$ is unblocked under $\succ$ if there does not exist a hospital $h$ and a nonempty set of contracts $Z \subseteq X_h \smallsetminus A$ such that $Z$ is contained in $h$'s most-preferred set from $Z \cup A$ and $Z_d \succeq_d Y_d$ for all doctors $d$ such that $Z_d \neq \varnothing$.

[14] Alkan and Gale (2003) introduced a related condition called *size monotonocity*.

# 4 Substitutable Completability

Standard many-to-one matching with contracts models impose a requirement that each hospital's preference relation be *many-to-one*, in the sense that a set of contracts is acceptable only if that set contains at most one contract with each doctor. However, as our Sherlock–Watson example in Section 2 illustrates, a hospital's many-to-one preference relation might reflect an underlying desire to assign a single doctor to multiple positions, even if the hospital is aware that the doctor demands at most one contract. In fact, we can think of a hospital's many-to-one preference relation as a projection of a more "complete" preference relation that ranks infeasible sets. An *extended preference relation* $\hat{\succ}_h$ is a strict linear order over *all* subsets of $X_h$; unlike preference relations, extended preference relations rank all subsets of $X_h$, not just feasible subsets.

**Definition 1.** A *completion* of a many-to-one preference relation $\succ_h$ of hospital $h$ is an extended preference relation $\hat{\succ}_h$ that agrees with $\succ_h$ on feasible subsets of $X_h$.

Effectively, $\hat{\succ}_h$ completes $\succ_h$ if we can obtain $\hat{\succ}_h$ by "inserting" infeasible sets into the linear order $\succ_h$. Equivalently, $\hat{\succ}_h$ completes $\succ_h$ if we can obtain $\succ_h$ as the projection of $\hat{\succ}_h$ to the many-to-one preference space; i.e., we obtain $\succ_h$ by restricting $\hat{\succ}_h$ to feasible sets. We say that a profile of extended preference relations $\hat{\succ}$ is a *completion* of a profile of preference relations $\succ$ if, for each hospital $h \in H$, the preference relation $\hat{\succ}_h$ is a completion of the associated preference relation $\succ_h$, and (by convention) $\hat{\succ}_d = \succ_d$ for each doctor $d \in D$.

The concept of stability naturally generalizes to extended preference relations: For an extended preference relation $\hat{\succ}$, the set of contracts $A$ is individually rational if $A_i$ is $i$'s most-preferred subset of $A_i$ under $\hat{\succ}_i$ for each agent $i$. Similarly, $A$ is unblocked under $\hat{\succ}_i$ if there is no hospital $h$ and set of contracts $Z$ such that $Z \mathrel{\hat{\succ}_h} A$ and each doctor $d$ associated with a contract in $Z$ (weakly) prefers (under $\hat{\succ}_d = \succ_d$) that contract to his outcome under $A$. Likewise, our substitutability and Law of Aggregate Demand conditions straightforwardly generalize to extended preference relations.

In general, we can construct nontrivial completions of a hospital's preferences by extending that hospital's preferences to incorporate at least one infeasible set. For example, consider the preference relation $\succ_h$ of hospital $h$ given in Section 2, which we reproduce here:

$$\{(s,h,r),(w,h,c)\} \succ_h \{(s,h,c)\} \succ_h \{(w,h,c)\} \succ_h \{(s,h,r)\} \succ_h \varnothing. \tag{1}$$

A natural completion of these preferences is the extended preference relation $\hat{\succ}_h$ constructed in Section 2,

$$\{(s,h,r),(s,h,c)\} \mathrel{\hat{\succ}_h} \{(s,h,r),(w,h,c)\} \mathrel{\hat{\succ}_h} \{(s,h,c)\} \mathrel{\hat{\succ}_h} \{(w,h,c)\} \mathrel{\hat{\succ}_h} \{(s,h,r)\} \mathrel{\hat{\succ}_h} \varnothing. \tag{2}$$

Looking at substitutable completions allows us to find stable outcomes under certain types of non-substitutable (many-to-one) preference relations. To see this, we begin by relating the set of outcomes stable under a preference profile to the set of outcomes stable under a completion of that profile.

**Lemma 1.** *Let $\hat{\succ}$ be a completion of the preference profile $\succ$. Then an outcome is stable with respect to $\succ$ if and only if that outcome is stable with respect to $\hat{\succ}$.*

*Proof.* Observe that if $A$ is a stable outcome under $\hat{\succ}$, then $A$ is individually rational for each doctor under $\hat{\succ}_D = \succ_D$. Furthermore, since doctors have unit demand, the individual rationality of $A$ for each doctor implies that $A$ contains at most one contract associated with each doctor. Consequently, for each hospital $h$, we have that $A_h$ contains at most one contract with each doctor—and thus is feasible. The individual rationality of $A$ for each hospital $h$ under the preferences $\succ$ then follows from the individual rationality of $A$ for each hospital $h$ under the preferences $\hat{\succ}$, as $\succ$ and $\hat{\succ}$ coincide for feasible sets. Finally, if $A$ were blocked under $\succ$, then there would exist a hospital $h$ and a (feasible) set of contracts $Z$ such that $Z \succ_h A_h$ and $Z_d \succeq_d A_d$ for all doctors $d$ such that $Z_d \neq \varnothing$. But as $\hat{\succ}$ is a completion of $\succ$, we would then immediately have that $A$ is blocked under $\hat{\succ}$, as $\succ$ and $\hat{\succ}$ coincide for feasible sets. Hence, $A$ is unblocked under $\succ$. Thus, we see that $A$ must be stable with respect to $\succ$.

Now, if $A$ is a stable outcome under $\succ$, then $A$ is individually rational for both doctors and hospitals under $\succ$, and so it is immediate that $A$ is individually rational for both doctors and hospitals under $\hat{\succ}$. Moreover, if $A$ were blocked under $\hat{\succ}$, then then there would exist a hospital $h$ and a set of contracts $Z$ such that $Z \mathrel{\hat{\succ}_h} A_h$ and $Z_d \mathrel{\hat{\succeq}_d} A_d$ for all doctors $d$ such that $Z_d \neq \varnothing$; note that $Z$ is feasible as each doctor $d$ (weakly) prefers $Z_d$ to the acceptable set $A_d$. But as $\hat{\succ}$ is a completion of $\succ$, we would then immediately have that $A$ is blocked under $\succ$, as $\succ$ and $\hat{\succ}$ coincide for feasible sets; thus $A$ is unblocked under $\hat{\succ}$. Hence, $A$ is stable under $\hat{\succ}$. $\square$

If a preference relation $\succ_h$ has a completion that is substitutable, then we say that $\succ_h$ is *substitutably completable*.[15] If every preference relation in a preference profile $\succ$ is substitutably completable, then we say that $\succ$ is *substitutably completable*. The remainder of this section demonstrates that substitutably completable preference relations inherit many useful properties from their completions.

In essence, we can think of a substitutably completable preference relation $\succ_h$ as the many-to-one projection of a substitutable (extended) preference relation $\hat{\succ}_h$ from a richer preference space. Indeed, we may view the completion $\hat{\succ}_h$ of a many-to-one preference relation $\succ_h$ as a preference relation in the setting of *many-to-many matching with contracts*, in which doctors, as well as hospitals, may demand multiple contracts. As substitutable preferences are sufficient to guarantee the existence of stable outcomes in many-to-many matching with contracts (Hatfield and Kominers, 2012, 2017), we see that substitutable completability is sufficient to guarantee the existence of a stable outcome:[16] If $\hat{\succ}$ is a substitutable completion

[15] Any substitutable preference relation $\succ_h$ is straightforwardly substitutably completable by taking the completion $\hat{\succ}_h$ to coincide with $\succ_h$ on feasible sets and having $\hat{\succ}_h$ rank all infeasible sets as unacceptable.

[16] In our work on contract language design in many-to-many matching with contacts (Hatfield and Kominers, 2017), we consider a many-to-many matching model in which each pair of agents is allowed to sign multiple contracts with each other. We argue there that allowing two given agents to sign multiple contracts with each other is valuable for modeling *many-to-many* matching with contracts, in part because it enables substitutable representations of some types of preferences. Our exercise here is different, however: we combine our substitutable completability insight with the Hatfield and Kominers (2017) existence result on stable outcomes in many-to-many matching with contracts settings in order to find stable outcomes in some *many-to-one* matching with contracts markets in which agents have preference complementarities.

of a preference profile $\succ$, then there exists at least one outcome that is stable with respect to $\hat{\succ}$—and hence, stable with respect to $\succ$ by Lemma 1.[17]

**Theorem 1.** *If a preference profile $\succ$ has a substitutable completion, then there exists an outcome that is stable with respect to $\succ$.*

In most matching settings, complementarities thwart stability—that is, stable outcomes can only be guaranteed when all agents have substitutable preferences. Indeed, in their original paper on many-to-one matching with contracts, Hatfield and Milgrom (2005, p. 921) had claimed that substitutability is necessary to ensure the existence of stable outcomes in many-to-one matching with contracts. Yet Theorem 1 shows that any substitutably completable preference profile is guaranteed to produce stable outcomes, even if that preference profile involves complementarities among contracts; as this suggests, Hatfield and Milgrom's claim was incorrect. Hatfield and Milgrom's error was first noted by Hatfield and Kojima (2008, 2010), but our work provides a new, intuitive explanation of what is going on: there may be non-substitutable preferences that nevertheless have a hidden, underlying substitutable structure that they inherit from associated, more complete, many-to-many preferences.

The Sherlock–Watson example of Section 2 illustrates the power of Theorem 1. Leveraging the substitutability of the completion $\hat{\succ}_h$ given by (2), Theorem 1 guarantees that there exists a stable outcome for any specification of Sherlock's and Watson's preferences.

However, as in the Sherlock–Watson example, there may be multiple stable outcomes

[17] In a sense, our work here hearkens back to the work of Fleiner (2003), who introduced a model of matching with contracts that does not sharply distinguish between many-to-one and many-to-many matching. Fleiner (2003) showed the existence of stable outcomes in his setting via Tarski's fixed-point theorem, building on an insight of Adachi (2000); in particular, Fleiner's (2003) work shows that the fixed-point approach does not depend, in principle, on whether hospitals are allowed to demand multiple contracts with a given doctor.

Our approach shows that passing between many-to-one and many-to-many matching is a helpful way to think about matching with contracts. However, our work also shows that simply treating the many-to-one model as a many-to-many model in which one side simply happens to have unit-demand preferences is an incomplete approach, as the many-to-one model has structure not present in the many-to-many model. Indeed, we sometimes need to transform hospitals' preference relations as we move from many-to-one matching with contracts to many-to-many matching with contracts in order to show the existence of stable outcomes. For instance, in a setting in which hospitals have preferences like those in the Sherlock–Watson example, results from many-to-many matching with contracts do not imply the existence of stable outcomes until hospitals' preferences are substitutably completed.

and, moreover, Theorem 1 does not provide a method for finding stable outcomes. Luckily, given a substitutable completion $\hat{\succ}$ of $\succ$, we can find a stable outcome using a well-known algorithm from matching theory: the *(doctor-proposing) cumulative offer process*. In the first step of the cumulative offer process, each doctor proposes his favorite contract according to $\succ$; each hospital then holds its most-preferred (under $\succ$) set of contracts from those proposed to it. In each subsequent round, each doctor not associated with a currently held contract proposes his most-preferred contract that has not yet been proposed (if any); each hospital then holds its most-preferred (under $\succ$) set of contracts from all the contracts that have been proposed to it so far. The algorithm ends when each doctor either is associated with a contract currently held by some hospital or has proposed every contract he finds acceptable; the outcome is the set of contracts held by the hospitals at that point.

Formally, the doctor-proposing cumulative offer process under $\succ$ proceeds as follows:[18]

**Step 1:** Each doctor *proposes* his most-preferred contract from $X$ under $\succ$ (assuming there is one); the set of proposed contracts is denoted $A^1$. Each hospital $h$ *holds* its most-preferred set of contracts from those that have been proposed; we call the set of all held contracts $Y^1$.

**Step $\tau$:** Each doctor not associated with a currently held contract, i.e., without a contract in $Y^{\tau-1}$ proposes his most-preferred contract that has not yet been proposed (if any), i.e., his most preferred contract from $X \setminus A^{\tau-1}$. If no contract is proposed, then the algorithm terminates and the outcome is the set of contracts held by the hospitals, $Y^{\tau-1}$. Otherwise, the set of contracts proposed in Steps 1 through $\tau$ is denoted $A^{\tau}$; each hospital $h$ holds its most-preferred set of contracts from $A^{\tau}$; the set of all held contracts is denoted $Y^{\tau}$. The algorithm then proceeds to Step $\tau + 1$.

Note that the cumulative offer process in principle allows hospitals to hold contracts from $A^{\tau}$ that are not held in a prior step $\hat{\tau}$ of the algorithm (where $\hat{\tau} < \tau$). However, when all

[18]The cumulative offer process was first introduced by Kelso and Crawford (1982) in a many-to-one matching with salaries model.

hospitals' preference relations are substitutable, hospitals never "take back" contracts that were available but not held at some prior step—precisely because substitutability rules out the possibility that a new contract offer makes a previously-rejected contract desirable. The fact that hospitals never take back contracts is key to the functioning of the cumulative offer process, as it ensures that no doctor is associated with more than one contract at the end of the algorithm: If some hospital $h$ had non-substitutable preferences, then $h$ might reject the contract $z$ at some step of the algorithm but then hold onto $z$ at the final step (after receiving other offers); if the doctor associated with $z$ made an offer to some other hospital which that hospital also held, then the doctor associated with $z$ would be associated with two contracts at the end of the cumulative offer process and so the outcome would be infeasible. Thus, substitutability ensures that the outcome of the cumulative offer process is feasible.

As the logic of the previous paragraph only depends on substitutability, it is also true for any substitutable completion $\hat{\succ}$ of $\succ$ that $\hat{\succ}$ never "takes back" any contract that is available but not held at some prior step. Moreover, since the doctors who propose in any given step are a subset of those doctors without a held contract, at every step of a cumulative offer process under a substitutable completion $\hat{\succ}$, each hospital holds at most one contract with each doctor; hence, by the definition of a completion, at every step the behavior under $\hat{\succ}$ is the same as the behavior under $\succ$. This logic implies that the path of the cumulative offer process under any substitutable completion $\hat{\succ}$ is the same as the path of the cumulative offer process under $\succ$.[19]

Meanwhile, as demonstrated by Hatfield and Kominers (2012, 2017), when all agents' preference relations are substitutable, the cumulative offer process yields a stable outcome. The preceding observations imply our next result.

**Theorem 2.** *If the preference profile $\succ$ has a substitutable completion, then the outcome*

[19] Additionally, when all hospitals' preference relations are substitutably completable, the doctor-proposing cumulative offer process is equivalent to a *(doctor-proposing) deferred acceptance process* (Gale and Shapley, 1962) under which, in Step $\tau$, each hospital is allowed to hold only contracts that were either held by that hospital in Step $(\tau - 1)$ or newly proposed in Step $\tau$. For completeness, we give a formal description of the deferred acceptance process in Appendix C.

*of the doctor-proposing cumulative offer process under $\succ$ is the same as the outcome of the doctor-proposing cumulative offer process under any substitutable completion of $\succ$; moreover, that outcome is stable under $\succ$.*

Theorem 2 implies that when we know that hospitals' preference relations are substitutably completable, we do not even need to compute substitutable completions in order to find stable outcomes—it is sufficient to run the cumulative offer process using the original preference profile.[20]

Another consequence of substitutable completability is that under the Law of Aggregate Demand, the cumulative offer process makes truth-telling a dominant strategy, just as it does under substitutable preferences (Hatfield and Milgrom, 2005). We say that the cumulative offer process is *strategy-proof (for doctors)* if no doctor can obtain a strictly-preferred outcome by misreporting his preference relation; that is, each doctor $d$ weakly prefers (under $\succ_d$) the contract he obtains (if any) in the outcome generated by the cumulative offer process under the profile $\succ$ to the contract he obtains (if any) in the outcome generated by the cumulative offer process when $d$ submits alternative preferences $\succ'_d$.

**Theorem 3.** *If, for each $h \in H$, the preference relation $\succ_h$ has a substitutable completion that satisfies the Law of Aggregate Demand, then the cumulative offer process is strategy-proof (for doctors).*[21,22]

# 5 Applications of Substitutable Completability

Since we first circulated the ideas described herein, a number of authors have developed applications of substitutable completability to real-world matching problems. Here, we briefly

[20]Additionally, Theorem 2 implies that the outcome of the cumulative offer process is, in a sense, canonical: If $\succ$ has a substitutable completion, then the cumulative offer process produces the same outcome regardless of which substitutable completion we use—and, moreover, that same outcome is produced by running the cumulative offer process using the original preference profile $\succ$.

[21]Indeed, our proof of Theorem 3 shows a stronger result: under the assumptions of Theorem 3, the cumulative offer process is *group strategy-proof (for doctors)*, in the sense that no coalition of doctors can make each doctor in the coalition strictly better off by jointly misreporting their choice functions.

[22]We discuss the relationship between substitutable completablity and classical structural results (such as lattice structure and the rural hospitals theorem) in Appendix A.4.

survey those applications, and also present a new application that generalizes the slot-specific priorities framework of Kominers and Sönmez (2015).

## 5.1 The Israeli Psychology Masters Match

In 2014, Hassidim et al. (2016a,b, 2017) redesigned the Israeli Psychology Masters Match (IPMM), which assigns approximately 1,400 applicants to approximately 600 positions in graduate programs in psychology in Israel each year. The principal goal of the IPMM redesign was to implement a mechanism that is stable and strategy-proof (for applicants).[23] However, the IPMM features a large range of contractual terms (such as type of degree and fellowship status), and some programs have complex preferences (such as affirmative action constraints and rules for balancing the allocation between clinical and research positions); as a result, many of the graduate programs involved in the IPMM have preferences that are not only non-substitutable, but also fail all the weakened substitutability conditions introduced prior to our work (see Hassidim et al. (2016a)). Nevertheless, in order to get the graduate programs to agree to the IPMM redesign, it was essential for the redesigned mechanism to enable programs to express preferences at their true levels of complexity. Hassidim et al. (2016a) were able to show (*after* soliciting unrestricted preference structures from the programs) that all the programs' preference structures are, in fact, substitutably completable.[24] Our results here were then used to facilitate stable and strategy-proof matching in the IPMM. The completion-based IPMM has now been successfully run for five years, with both programs and students expressing satisfaction with the process (Hassidim et al., 2016a).

[23]Stability was desired in order to eliminate "unraveling" of the type observed by Roth and Xing (1994)—there were widespread beliefs that some departments coordinated amongst themselves on who would make offers to which candidates, and that other departments made more offers than they had positions available (in order to ensure they filled their quota). Roth (1991) showed that stable mechanisms have alleviated unraveling in the United States and elsewhere. Strategy-proofness was desired to simplify the strategic problem faced by applicants (Hassidim et al., 2016b).

[24]Moreover, the natural substitutable completions that Hassidim et al. (2016a) identified satisfy the Law of Aggregate Demand.

## 5.2 College Admissions in India

Aygün and Turhan (2016, 2017) studied the allocation of over 300,000 students to the Indian Institutes of Technology (IIT). In the IIT matching mechanism, schools must set aside a certain number of slots for students from different privileged groups; however, a reserved slot may "revert" to a regular seat if it is not taken by a member of a privileged group.[25] A student from a privileged group may prefer a seat reserved for privileged groups (as such seats come with significant financial aid) but also might prefer an unreserved seat (as students who take reserved seats face discrimination on campus). Aygün and Turhan (2016) observed that the choice procedures used in the IIT student matching mechanism do not generate substitutable preferences; moreover, those choice procedures are not examples of any of the non-substitutable, but still well-behaved, preference classes identified by previous work. Thus, Aygün and Turhan (2016) used our theory of substitutable completability (specifically, our Theorems 1–3) to argue that the IIT system could improve its allocation mechanism by using an implementation of the cumulative offer process.[26]

## 5.3 College Admissions with Multiple Offers

Yenmez (2018) buildt on our work here to propose a new approach for centralized college admissions. Yenmez (2018) treated college admissions as a many-to-many matching with contracts problem, in which students can be matched with many "admissions offers" which may include financial aid. Implementing binding "early decision" rules into college admissions introduces non-substitutabilities in colleges' preferences; however, Yenmez (2018) showed that every college's preferences have a substitutable completion. Yenmez (2018) then generalized our results to his many-to-many matching with contracts setting to show the existence of

[25]The privileged groups are comprised of "scheduled castes," "scheduled tribes," and "other backward classes," groups that have been historically disadvantaged in India.

[26]In related work, Aygün and Bó (2017) show that under the college admissions process in Brazil, which also incorporates affirmative action constraints, the preferences of colleges can be represented by a slot-specific preference structure (Kominers and Sönmez, 2016), which are a special case of our framework (see Section 5.6).

stable admissions outcomes.[27]

## 5.4 Matching with Budget Constraints

Hassidim et al. (2018) considered a model of college admissions matching with financial aid and budget constraints. In their setting, the terms of contract specify a financial aid offer (from a discrete set of possible offers); each college has a fixed financial aid budget and wants to recruit the best students that it can. It is well-known that budget constraints like those in the setting of Hassidim et al. (2018) induce non-substitutabilities in the preferences of colleges. However, Hassidim et al. (2018) proved that, nevertheless, colleges with budget constraints have substitutably completable preferences, and thus a stable outcome is guaranteed to exist in their setting. Indeed, their work generalizes to any setting (such as a labor market) in which "hospitals" have multiple positions and rank candidates according to a linear order, but only have a limited budget with which to recruit candidates. This is particularly surprising as budget constraints have proven difficult to incorporate into other market design settings, such as auctions (Che and Gale, 1998; Benoît and Krishna, 2001; Milgrom, 2004; Pai and Vohra, 2014).

## 5.5 Interdistrict School Choice

Recently, Hafalir et al. (2019) introduced a model of interdistrict school choice in which each student simultaneously participates in the school choice programs of multiple districts. Hafalir et al. (2019) use the matching with contracts framework to model interdistrict school choice; in their setup, students match with districts and each contract between a student and a district specifies which school that student attends. District preferences turn out to be non-substitutable for reasons quite similar to the Sherlock–Watson example: New applications to a given school may cause a district to reject other applications to that school

[27] For Yenmez's (2018) results, it is essential that students receive no more than one admissions offer from each college—a version of an assumption that Kominers (2012) called "unitarity." In other work (Hatfield and Kominers, 2017), we have shown that in non-unitary many-to-many matching with contracts models, substitutability is necessary (in the maximal domain sense) for the existence of stable outcomes. (See also Klaus and Walzl (2009) for a non-unitary model of many-to-many matching with contracts.)

it would have otherwise accepted; this makes it possible for the district to choose contracts with those newly-rejected students at other schools (even if the district previously rejected those contracts so as to avoid assigning any student to more than one school). Just like in the Sherlock–Watson example, where the hospital switches which division (research or clinical) it assigns Sherlock to as a function of whether Watson is available, the district may switch which school some student is assigned to as function of which other students are available. And, as in the Sherlock–Watson example, Hafalir et al. (2019) obtain a substitutable completion (that satisfies the Law of Aggregate Demand) of a district's preferences by imagining that multiple schools within the same district can take a contract with a given student; they thus find that stable and strategy-proof matching is possible in their setting.

## 5.6 Tasks-and-Slots Priorities

Kominers and Sönmez (2015) introduced *slot-specific priorities*, a general class of preference structures that can be used to incorporate diversity and other constraints into many-to-one matching with contracts. Under slot-specific priorities, each hospital has a set of *slots*, and each slot has its own preference ranking over contracts. Slots at a hospital are filled in sequence according to a *precedence order*. Kominers and Sönmez (2015) found that slot-specific priorities arise in a number of real-world settings, including cadet–branch matching (Sönmez and Switzer, 2013; Sönmez, 2013),[28] airline upgrade allocation (Kominers and Sönmez, 2015), and the design of affirmative action mechanisms (Kominers and Sönmez, 2015); Dimakopoulos and Heller (2019) have subsequently shown that the entry-level German labor market for lawyers is also well-modeled by slot-specific priorities.

[28]Building on our approach, Kojima et al. (2018) have shown another new way of obtaining the results of Sönmez and Switzer (2013): They consider the cadet–branch matching setting as a many-to-many matching model, and show that in that setting the branches' choice functions can be represented by $M^\natural$-concave functions. They then apply their Corollary 1 to show that the branches' "many-to-many" (or, equivalently, completed) choice functions are substitutable, satisfy the Law of Aggregate Demand, and satisfy the irrelevance of rejected contracts condition; hence, the cumulative offer process is stable and strategy-proof (see also Kamada and Kojima (2012, 2014, 2018)).

Kominers and Sönmez (2015) showed that the cumulative offer mechanism is stable and strategy-proof under slot-specific priorities using a complex argument based on constructing an auxiliary one-to-one matching with contracts economy. But as it happens, every slot-specific preference structure is substitutably completable. Indeed, in Appendix E, we introduce a new class of substitutably completable preference structures that generalizes slot-specific priorities. Under our *tasks-and-slots priorities*, hospitals have two different types of positions: *tasks* and *slots*.[29] "Task" positions are always filled before "slot" positions. The order in which tasks are filled may depend on the set of contracts available;[30] however, any two tasks either have identical preference orderings or find disjoint sets of contracts acceptable. Meanwhile, in principle, any contract can be accepted by any slot, but the sequence in which slots are filled can not depend on the set of contracts available.[31]

## 5.7 Unilaterally Substitutable Preferences

Hatfield and Kojima (2010) introduced *unilateral substitutability*, a condition on preferences that ensures that the cumulative offer process produces a stable outcome and is strategy-proof for doctors. Unilateral substitutability has been central in the analysis of cadet–branch matching problems: Sönmez and Switzer (2013) and Sönmez (2013) showed that U.S. military branches' preferences over contracts with cadets are unilaterally substitutable (but not substitutable), and then used this observation to show the existence of a stable and strategy-proof cadet–branch matching mechanism very similar to the mechanism already used by the U.S. Army.

In fact, any unilaterally substitutable preference relation is substitutably completable, as Kadam (2017) has recently shown. Thus, for many applications, substitutable completability

[29]The slot-specific class is recovered by taking the set of tasks to be empty.

[30]This property allows tasks-and-slots preference structures to fail the *irrelevance of rejected contracts* condition of Aygün and Sönmez (2013). However, as we show in Appendix E, every tasks-and-slots preference structure has a substitutable completion that *does* satisfy the irrelevance of rejected contracts condition (and the Law of Aggregate Demand). To our knowledge, tasks-and-slots preference structures are the first preference structures that fail the irrelevance of rejected contracts condition for which stable and strategy-proof matching can be guaranteed.

[31]In an earlier version of this paper (Hatfield and Kominers, 2018), we showed that the preferences of schools n the German teacher traineeship market fall within the class of tasks-and-slots preference structures.

may provide a technically simpler and more intuitive alternative to unilateral substitutability.[32]

# 6 Conclusion

Preferences that are substitutably completable have a hidden, underlying substitutable structure: they are effectively projections of substitutable preferences from the broader preference domain of many-to-many matching with contracts to the preference domain of many-to-one matching with contracts. Because of this structure, the existence of a substitutable completion (that satisfies the Law of Aggregate Demand) guarantees that the cumulative offer process produces a stable outcome and is strategy-proof for doctors.

In the Hatfield and Milgrom (2005) formulation of many-to-one matching with contracts, the condition that each doctor is assigned at most one contract is enforced by both restricting doctors to demand at most one contract and restricting hospitals to demand at most one contract with each doctor. However, the restriction on doctor preferences is sufficient to guarantee that each doctor has only one contract in any stable outcome; this implies that the restriction on hospital preferences is (formally) unnecessary. Thus, in some sense, our approach hearkens back to the earlier matching with contracts model of Fleiner (2003), which did not formally impose the constraint that each hospital can choose at most one contract with each doctor. Substitutable completability shows that this issue is not just a theoretical curiosity; rather, if we treat each hospital as willing to accept multiple contracts with the same doctor, then we can extend the applicability of the matching with contracts model. In particular, our theory of substitutable completability enables us to achieve stable and strategy-proof matching in settings ranging from matching with budget constraints

[32] Hatfield and Kojima (2010) also showed that stable outcomes are guaranteed to exist when hospitals' preferences satisfy a weaker condition called *bilateral substitutability*. In Appendix D, we show that there exist substitutably completable preferences that are not bilaterally substitutable (and, hence, do not satisfy the stronger condition of unilateral substitutability); there, we also show that there exist hospital preferences that are bilaterally substitutable but are not substitutably completable. It is an open question whether there is a condition on hospital preferences sufficient and necessary (in the maximal domain sense) to guarantee the existence of stable outcomes. In work subsequent to ours, Hatfield, Kominers, and Westkamp (2019) have identified a set of conditions that are both sufficient and necessary (in the maximal domain sense) for the guaranteed existence of stable and strategy-proof matching mechanisms.

(Hassidim et al., 2018) to college admissions (Aygün and Turhan, 2016; Yenmez, 2018).

Our results highlight how a deep understanding of substitutability is essential for market design. Matching with contracts depends crucially on substitutability, but recent work including ours and others' (e.g., Ostrovsky (2008), Milgrom (2009), Milgrom and Strulovici (2009), Echenique (2012), Ostrovsky and Paes Leme (2015), Kojima et al. (2018), and Jagadeesan (2019)) shows that substitutability is subtle—indeed, it sometimes hides in plain sight.

> *"Circumstantial evidence is a very tricky thing [. . . .] It may seem to point very straight to one thing, but if you shift your own point of view a little, you may find it pointing in an equally uncompromising manner to something entirely different."*
>
> —Sherlock Holmes, in *The Boscombe Valley Mystery*

# A Generalized Model and Results

In this appendix, we generalize the model in the main text to work with choice functions instead of preference relations. Working with choice functions necessitates using slight variations of our main concepts. We thus show linkages between these concepts in Appendices B.1 and B.2 and subsequently show in Appendix B.3 that the results we derive here imply the results in the main text.

## A.1 Choice Functions

Each agent $i$ has a *choice function* $C^i$ that specifies, for any given set of contracts $Y$, the set of contracts $i$ chooses from $Y$. We require that each agent $i$ only choose contracts he is associated with, i.e., $C^i(Y) \subseteq Y_i$. Moreover, doctors have *unit demand*, i.e., for all doctors $d$ and all sets of contracts $Y$, the set $C^d(Y)$ contains at most one contract, i.e., $|C^d(Y)| \leq 1$. Hospitals, meanwhile, may demand multiple contracts. We say that the choice function $C^h$ of a hospital is *many-to-one* if it only selects sets of contracts that contain at most one contract with each doctor (i.e., $|[C^h(Y)]_d| \leq 1$ for each $d \in D$). A *profile* of choice functions is a vector $C = (C^i)_{i \in D \cup H}$.

Except where explicitly noted otherwise, we only consider choice functions that satisfy the *irrelevance of rejected contracts* condition of Aygün and Sönmez (2013, 2014); this condition is an "independence of irrelevant alternatives" condition requiring that the set of contracts an agent chooses does not change when that agent loses access to a contract not in that chosen set.[33] Formally, a choice function $C^i$ satisfies the irrelevance of rejected contracts condition if, for all $Y \subseteq X$ and $z \in X \smallsetminus Y$, whenever $z \notin C^i(Y \cup \{z\})$, we have $C^i(Y \cup \{z\}) = C^i(Y)$. We say that a profile of choice functions $C$ satisfies the irrelevance of rejected contracts condition if $C^h$ satisfies the irrelevance of rejected contracts condition for each $h \in H$.

A choice function $C^i$ is *substitutable* if for all $x, z \in X$ and $Y \subseteq X$, if $z \notin C^i(Y \cup \{z\})$, then $z \notin C^i(\{x\} \cup Y \cup \{z\})$. In our framework, doctors' choice functions are always substitutable

[33] In particular, we assume throughout that all doctors' choice functions satisfy the irrelevance of rejected contracts condition.

because doctors have unit demand.[34,35] A choice function $C^i$ satisfies the *Law of Aggregate Demand* if for all $\hat{Y} \subseteq Y \subseteq X$, we have $|C^i(\hat{Y})| \leq |C^i(Y)|$.

Our choice function formulation extends and generalizes the model in the main text. In particular, any (extended) preference relation $\succ_i$ for $i$ naturally *induces* a choice function $C^i$, under which $i$ chooses the subset of $Y$ that is highest-ranked according to the preference relation $\succ_i$; that is,

$$C^i(Y) = \max_{\succ_i}\{Z \subseteq X_i : Z \subseteq Y\}, \tag{3}$$

where $\max_{\succ_i}$ indicates maximization with respect to the ordering $\succ_i$.

## A.2 Stability under Choice Functions

Here, we define *(choice-theoretic) stability* in the usual way (see, e.g., Hatfield and Milgrom (2005)); for simplicity, we shall refer to choice-theoretic stability as stability throughout Appendix A. An outcome is *(choice-theoretic) stable* if it is both *(choice-theoretic) individually rational* and *(choice-theoretic) unblocked*:

- An outcome is *individually rational* if no agent wishes to unilaterally abrogate any of his contracts in $A$; formally, $A$ is individually rational under $C$ if $C^i(A) = A_i$ for all $i \in D \cup H$.

- For a hospital $h$, we say that a nonempty set $Z$ is a *block* for $A$ if $Z \subseteq X_h \smallsetminus A$ such that $Z_i \subseteq C^i(A \cup Z)$ for all $i$ associated with contracts in $Z$. An outcome $A$ is *unblocked* if there is no block for $A$, i.e., no hospital and set of doctors can improve upon $A$ for themselves by negotiating new contracts outside of $A$ (while possibly dropping some of the contracts in $A$).

[34]Note that this observation also depends crucially on our assumption that doctors' choice functions satisfy the irrelevance of rejected contracts condition.

[35]A choice function $C^h$ satisfies both the substitutability condition and the irrelevance of rejected contracts condition if and only if it is *path-independent*, i.e., if for every $Y, Z \subseteq X$, we have that $C^h(Y \cup Z) = C^h(Y \cup C^h(Z))$. The linkage between path independence and our key conditions was first noted by Aizerman and Malishevski (1981); Chambers and Yenmez (2017, 2018) recently extended the observation to matching with contracts.

## A.3 Substitutable Completability

We now generalize the concept of completion to choice functions. Like our preference-based definition, our concept of completion allows hospitals to choose infeasible sets, while requiring consistency with the original preferences (here, choice functions) whenever feasible sets are chosen.

**Definition 2.** A *completion* of a many-to-one choice function $C^h$ of hospital $h \in H$ is a choice function $\hat{C}^h$ such that for all $Y \subseteq X$, either

- $\hat{C}^h(Y) = C^h(Y)$, or
- $\hat{C}^h(Y)$ is infeasible, i.e., $z, \hat{z} \in [\hat{C}^h(Y)]_d$ for some $d \in D$.

We say that a profile of choice functions $\hat{C}$ is a *completion* of a profile of choice functions $C$ if, for each hospital $h \in H$, the choice function $\hat{C}^h$ is a completion of the associated choice function $C^h$, and $\hat{C}^d = C^d$ for each doctor $d \in D$. Note that every choice function is a completion of itself.

We now state a version of Lemma 1 in the language of choice functions.[36]

**Lemma A.1.** *If $\hat{C}$ is a completion of a profile of choice functions $C$, and $\hat{C}$ satisfies the irrelevance of rejected contracts condition, then any outcome stable with respect to $\hat{C}$ is stable with respect to $C$.*

*Proof.* We assume that $A$ is stable with respect to $\hat{C}$, and show that $A$ is stable with respect to $C$. We prove the result in three steps:

**$A$ is individually rational for doctors under $C$:** As doctors have the same choice functions under $\hat{C}$ as under $C$, the individual rationality of $A$ under $\hat{C}^d$ for each doctor

[36] Note that, under the concept of stability introduced in Section 3, an outcome is stable under $\succ$ if and only if it is stable under a completion $\hat{\succ}$ of $\succ$. We do not have such an equivalence for choice functions, since an outcome can be choice-theoretic stable under the choice function induced by a preference relation $\succ$ but not be choice-theoretic stable under the choice function induced by the preference relation $\hat{\succ}$ which completes $\succ$ (see the discussion after Proposition B.5).

$d \in D$ immediately implies the individually rationality of $A$ under $C^d$ for each doctor $d \in D$.

$A$ **is individually rational for hospitals under** $C$**:** The individual rationality of $A$ for doctors implies that each doctor has at most one contract in $A$, i.e., $|A_d| \leq 1$ for each $d \in D$. Then, as $\hat{C}^h$ completes $C^h$, it follows that $C^h(A) = \hat{C}^h(A)$ for all $h \in H$ as $A$ does not contain two (or more) contracts with any individual doctor; hence, the individual rationality of $A$ under $\hat{C}^h$ for each hospital $h \in H$ immediately implies the individually rationality of $A$ under $C^h$ for each hospital $h \in H$.

$A$ **is unblocked under** $C$**:** Suppose that $A$ is blocked under $C$ by some hospital $h$ and a blocking set $Z \subseteq X_h \smallsetminus A$ under $C$. First, as $Z$ blocks $A$ under $C$, and $\hat{C}^d = C^d$ for each $d \in D$, we know that

$$Z_d \subseteq C^d(Z \cup A) = \hat{C}^d(Z \cup A) \quad \text{for all } d \in D. \tag{4}$$

Now, as $\hat{C}^h$ completes $C^h$, we know from the definition of completability that either

- $\hat{C}^h(Z \cup A) = C^h(Z \cup A)$, or
- there exist distinct $z, \hat{z} \in W \equiv \hat{C}^h(Z \cup A)$ such that $\mathsf{d}(z) = \mathsf{d}(\hat{z})$.

In the former case, we have $Z_h \subseteq C^h(Z \cup A) = \hat{C}^h(Z \cup A)$, as $Z$ blocks $A$ under $C$; combining this fact with (4) shows that $Z$ blocks $A$ under $\hat{C}$, contradicting the stability of $A$ under $\hat{C}$.

In the latter case, we note that as $A$ is individually rational for doctors under $C$, we must have $|A_{\mathsf{d}(z)}| \leq 1$ for each $d \in D$. Then, as we have $z, \hat{z} \in W = \hat{C}^h(Z \cup A)$ such that $\mathsf{d}(z) = \mathsf{d}(\hat{z})$, we know that $\bar{Z} \equiv W \smallsetminus A$ must be nonempty. Now, we have

$$\hat{C}^h(\bar{Z} \cup A) = \hat{C}^h((W \smallsetminus A) \cup A) = \hat{C}^h(W \cup A) = \hat{C}^h((Z \cup A) \cup A) = \hat{C}^h(Z \cup A) = W \supseteq \bar{Z}, \tag{5}$$

where the third equality follows from the fact that $\hat{C}^h$ satisfies the irrelevance of rejected contracts condition and $W = \hat{C}^h(Z \cup A)$. Combining (5) with (4) (for the $d \in \mathsf{d}(\bar{Z}) \subseteq D$) shows that $\bar{Z}$ blocks $A$ under $\hat{C}$, contradicting the stability of $A$ under $\hat{C}$.

The preceding three observations show that $A$ is stable with respect to $C$. $\square$

If a choice function $C^h$ has a completion that is substitutable, we say that $C^h$ is *substitutably completable*.[37] If every choice function in a profile of choice functions $C$ is substitutably completable, then we say that $C$ is *substitutably completable*.

We now show that for any profile of substitutably completable choice functions $C$, there exists a stable outcome.

**Theorem A.1.** *If the profile of choice functions $C$ has a substitutable completion that satisfies the irrelevance of rejected contracts condition, then there exists an outcome that is stable with respect to $C$.*

*Proof.* Let $\hat{C}$ be a substitutable completion for $C$. By Theorem 3 of Hatfield and Kominers (2012), the (generalized) doctor-proposing cumulative offer process of Hatfield and Milgrom (2005) yields a (many-to-many) matching outcome $A$ that is stable with respect to $\hat{C}$. By Lemma A.1, $A$ is stable with respect to $C$. $\square$

Although Lemma A.1 shows that any outcome stable under a completion of $C$ must also be stable under $C$, different completions of a choice function $C$ may yield different sets of stable outcomes.

**Example A.1.** Let $H = \{h\}$, $D = \{d, e\}$, and $X = \{x, \hat{x}, y, \hat{y}\}$ where $x$ and $\hat{x}$ are associated with $d$ and $h$ and $y$ and $\hat{y}$ are associated with $e$ and $h$. We consider the hospital choice function $C^h$ induced by the preference relation

$$\{x, \hat{y}\} \succ_h \{\hat{x}, y\} \succ_h \{\hat{x}, \hat{y}\} \succ_h \{x, y\} \succ_h \{\hat{x}\} \succ_h \{\hat{y}\} \succ_h \{x\} \succ_h \{y\} \succ \varnothing,$$

[37] Note that as every choice function is a completion of itself, all substitutable choice functions are trivially substitutably completable.

along with choice functions $C^d$ and $C^e$ respectively induced by the preference relations

$$\{x\} \succ_d \{\hat{x}\} \succ_d \varnothing$$

$$\{y\} \succ_e \{\hat{y}\} \succ_e \varnothing.$$

There are three outcomes stable under $C$: $\{x, \hat{y}\}$, $\{\hat{x}, y\}$, and $\{x, y\}$.

Additionally, there are two different substitutable completions of $C^h$, induced respectively by the extended preference relations

$$\{y, \hat{y}\} \mathrel{\hat{\succ}_h} \{x, \hat{y}\} \mathrel{\hat{\succ}_h} \{\hat{x}, y\} \mathrel{\hat{\succ}_h} \{\hat{x}, \hat{y}\} \mathrel{\hat{\succ}_h} \{x, y\} \mathrel{\hat{\succ}_h} \{\hat{x}\} \mathrel{\hat{\succ}_h} \{\hat{y}\} \mathrel{\hat{\succ}_h} \{x\} \mathrel{\hat{\succ}_h} \{y\} \mathrel{\hat{\succ}_h} \varnothing \text{ and}$$

$$\{x, \hat{x}\} \mathrel{\hat{\succ}'_h} \{x, \hat{y}\} \mathrel{\hat{\succ}'_h} \{\hat{x}, y\} \mathrel{\hat{\succ}'_h} \{\hat{x}, \hat{y}\} \mathrel{\hat{\succ}'_h} \{x, y\} \mathrel{\hat{\succ}'_h} \{\hat{x}\} \mathrel{\hat{\succ}'_h} \{\hat{y}\} \mathrel{\hat{\succ}'_h} \{x\} \mathrel{\hat{\succ}'_h} \{y\} \mathrel{\hat{\succ}'_h} \varnothing.$$

The completed choice profiles induced by these extended preference relations yield different sets of stable outcomes: $\{\hat{x}, y\}$ and $\{x, y\}$ are stable under the first, while $\{x, \hat{y}\}$ and $\{x, y\}$ are stable under the second.

That said, there is a distinguished outcome that is stable under *every* completion of $C$: the result of the *(doctor-proposing) cumulative offer process*.

We now generalize the cumulative offer process to the setting with choice functions. For a set of contracts $Y$, we say that $x$ is the *most-preferred* contract from $Y$ for $d$ under $C$ if $d$ chooses $\{x\}$ from $Y$ under $C$, i.e., $\{x\} = C^d(Y)$; with this definition, the doctor-proposing cumulative offer process proceeds analogously to the cumulative offer process as stated in the main text. For completeness, we state the full algorithm in terms of choice functions here.

**Step 1:** Each doctor *proposes* his most-preferred contract from $X$ under $C$ (assuming there is one); the set of proposed contracts is denoted $A^1$. Each hospital $h$ *holds* its favorite set of contracts from those that have been proposed, i.e., $C^h(A^1)$.

**Step $\tau$:** Each doctor not associated with a currently held contract proposes his most-preferred contract that has not yet been proposed (if any), i.e., his most preferred contract from $X \smallsetminus A^{\tau-1}$ under $C$. If no contract is proposed, then the algorithm terminates and the outcome is the set of contracts held by the hospitals from the set of proposed contracts, i.e., $\bigcup_{h \in H} C^h(A^{\tau-1})$. Otherwise, the set of contracts proposed in Steps 1 through $\tau$ is denoted $A^\tau$; each hospital $h$ holds its favorite set of contracts from those that have been proposed, i.e., $C^h(A^\tau)$; and the algorithm proceeds to Step $\tau + 1$.

It is immediate that the cumulative offer process under $\succ$ proceeds identically to the cumulative offer process under the profile of choice functions induced by $\succ$; in particular, $A^\tau$ is the same at each step $\tau$.

Meanwhile, when all agents' choice functions are substitutable (and satisfy the irrelevance of rejected contracts condition), the cumulative offer process yields a stable outcome (Hatfield and Kominers, 2012, 2017). The preceding observations imply our next result, which generalizes Theorem 2 to the setting of choice functions.

**Theorem A.2.** *If $\hat{C}$ is a substitutable completion of $C$, then the outcome of the doctor-proposing cumulative offer process under $\hat{C}$ is the same as the outcome of the doctor-proposing cumulative offer process under $C$; moreover, if $\hat{C}$ satisfies the irrelevance of rejected contracts condition, that outcome is stable under $C$.*

*Proof.* We fix a profile of choice functions $\hat{C}$ that substitutably completes $C$. We show by induction that the cumulative offer process under $\hat{C}$ corresponds step-by-step to the cumulative offer process under $C$; it follows immediately that those processes then have the same outcome.

Let $A^\tau$ be the set of available contracts at the end of Step $\tau$ of the cumulative offer process under $C$; similarly, let $\hat{A}^\tau$ be the set of available contracts at the end of Step $\tau$ of the cumulative offer process under $\hat{C}$. Our inductive hypotheses are that

1. $A^\tau = \hat{A}^\tau$ and

2. at each Step $\tau$, we have, for each $h \in H$, that $C^h(A^\tau) = \hat{C}^h(\hat{A}^\tau)$.

It follows immediately from the definition of the cumulative offer process that $A^1 = \hat{A}^1$.

Moreover, since $A^1 = \hat{A}^1$ has at most one contract with each doctor, $C^h(A^1) = \hat{C}^h(A^1) = \hat{C}^h(\hat{A}^1)$ for all $h \in H$; therefore, the second inductive hypothesis is also satisfied at $\tau = 1$.

Hence, we suppose that $A^{\tau-1} = \hat{A}^{\tau-1}$ and suppose that for each $h \in H$, we have $C^h(A^{\tau-1}) = \hat{C}^h(\hat{A}^{\tau-1})$. By construction, then, the same set of doctors is held at the beginning of Step $\tau$ of both the cumulative offer process under $C$ and the cumulative offer process under $\hat{C}$; hence, the same set of doctors makes proposals in Step $\tau$ of both processes. Moreover, since $A^{\tau-1} = \hat{A}^{\tau-1}$ (i.e., the same sets of contracts have been proposed prior to Step $\tau$), we know that each doctor proposing in Step $\tau$ proposes the same contract in both cumulative offer processes. Consequently, we see immediately that $A^\tau = \hat{A}^\tau$.

Now, since $A^\tau = \hat{A}^\tau$, we have that $\hat{C}^h(\hat{A}^\tau) = \hat{C}^h(A^\tau)$. To prove the second inductive hypothesis, suppose that

$$\hat{C}^h(\hat{A}^\tau) = \hat{C}^h(A^\tau) \neq C^h(A^\tau),$$

seeking a contradiction. Since $\hat{C}^h$ completes $C^h$, there exists $z, \hat{z} \in \hat{C}^h(A^\tau)$ such that $\mathsf{d}(z) = \mathsf{d}(\hat{z})$. Now, we can not have $\{z, \hat{z}\} \subseteq \hat{C}^h(A^{\tau-1})$, as $C^h(A^{\tau-1})$ contains at most one contract with each doctor, and $\hat{C}^h(A^{\tau-1}) = \hat{C}^h(\hat{A}^{\tau-1}) = C^h(A^{\tau-1})$ by the second inductive hypothesis. Thus, without loss of generality, we have $z \notin \hat{C}^h(A^{\tau-1})$. But then, we have a contradiction to the substitutability of $\hat{C}^h$, as $z \notin \hat{C}^h(A^{\tau-1})$, but $z \in \hat{C}^h(A^\tau)$, and $A^{\tau-1} \subseteq A^\tau$. The preceding argument shows the first half of the theorem.

To prove the latter half of the theorem, note that if $\hat{C}$ is a substitutable completion of $C$ that satisfies the irrelevance of rejected contracts condition, then Theorem 3 of Hatfield and Kominers (2012) implies that the outcome $Y$ of the cumulative offer process under $\hat{C}$ is stable with respect to $\hat{C}$; Lemma A.1 then implies that $Y$ is stable under $C$, as well. □

Theorem A.2 implies that when hospitals' choice functions are substitutably completable in a way that satisfies the irrelevance of rejected contracts condition, we do not need to compute substitutable completions in order to find stable outcomes—it is sufficient to run the cumulative offer process using the hospitals' original choice functions. Indeed, if $C$ is substitutably completable, then the cumulative offer process and the (doctor-proposing) deferred acceptance algorithm coincide:[38] At each step of the cumulative offer process, each hospital will choose a set of contracts composed only of contracts it held previously and contracts newly offered to it, as the hospital's choice function is substitutable and satisfies the irrelevance of rejected contracts condition. Thus, the fact that previously-rejected contracts are unavailable to a hospital under the deferred acceptance process is irrelevant.

Finally, we generalize Theorem 3 to the setting with choice functions; along the way, we show what is fact in a stronger conclusion, that the cumulative offer process is *group strategy-proof (for doctors)*, in the sense that no coalition of doctors can make each doctor in the coalition strictly better off by jointly misreporting their choice functions.[39]

**Theorem A.3.** *If, for each $h \in H$, the choice function $C^h$ has a substitutable completion that satisfies the Law of Aggregate Demand and the irrelevance of rejected contracts condition, then the cumulative offer process is group strategy-proof.*

*Proof.* Consider any substitutable completion $\hat{C}$ of $C$ such that $\hat{C}^h$ satisfies the Law of Aggregate Demand (and the irrelevance of rejected contracts condition) for each $h \in H$. As the doctor-proposing deferred acceptance mechanism selects the doctor-optimal stable outcome under the completed choice profile $\hat{C}$, it follows from Theorem 10 of Hatfield and Kominers (2012) (which extends the result of Hatfield and Kojima (2009) to the setting of matching in networks) that the doctor-proposing deferred acceptance mechanism is (group) strategy-proof for doctors. □

[38] We formally define the deferred acceptance process for preference relations in Appendix C.

[39] Formally, a mechanism (such as the cumulative offer process) is *group strategy-proof (for doctors)* if, for any choice function profile $C$ and set of doctors $\tilde{D} \subseteq D$, there is no alternative choice function profile $(\tilde{C}^{\tilde{D}}, C^{D \smallsetminus \tilde{D}}, C^H)$ such that every doctor in $\tilde{D}$ strictly prefers the outcome of the mechanism under $(\tilde{C}^{\tilde{D}}, C^{D \smallsetminus \tilde{D}}, C^H)$ to the outcome of the mechanism under $C$.

## A.4 The Structure of the Set of Stable Outcomes under Substitutable Completability

In matching with contracts, when choice functions are substitutable, there always exists a doctor-optimal stable outcome, i.e., a stable outcome that all doctors (weakly) prefer to every other stable outcome (see, e.g., Hatfield and Milgrom (2005)); in fact, this is the outcome selected by the doctor-proposing cumulative offer process. However, when choice functions are substitutably completable, the outcome selected by the cumulative offer process may not be doctor-optimal among outcomes stable under the original preferences; in fact, there may not exist a doctor-optimal stable outcome under the original choice functions. For instance, in our Sherlock–Watson example, Sherlock prefers the stable outcome $\{(s, h, c)\}$, while Watson prefers the other stable outcome $\{(s, h, r), (w, h, c)\}$.[40]

Like most arguments for strategy-proofness, Theorem A.3 uses a form of the *rural hospitals theorem* (Roth, 1984; Hatfield and Milgrom, 2005), which states that, when choice functions are substitutable and satisfy the Law of Aggregate Demand (and the irrelevance of rejected contracts condition), the number of contracts each agent signs is invariant across stable outcomes. However, while the rural hospitals theorem applies to any fixed substitutable completion that satisfies the Law of Aggregate Demand, its conclusion may not hold under the original choice profile $C$.[41] For instance, in the Sherlock–Watson example of Section 2, there are two stable outcomes with different numbers of contracts, even though $\succ$ (and its induced profile of choice functions $C$) satisfies the Law of Aggregate Demand itself and has a substitutable completion that satisfies the Law of Aggregate Demand.

[40] The Sherlock–Watson example also shows that the set of stable outcomes under a substitutably completable preference profile need not form a lattice in the usual way, as Sherlock's and Watson's preferences over stable outcomes are not aligned.

[41] However, Theorem 2 implies that the rural hospitals theorem holds across completions, in the sense that the number of contracts each agent signs is invariant across outcomes that are stable under some substitutable completion that satisfies the Law of Aggregate Demand (and the irrelevance of rejected contracts condition).

# B Linkages between Choice Functions and Preference Relations

## B.1 The Relationship between Choice Functions and Preference Relations

Recall that any (extended) preference relation $\succ_i$ for $i$ naturally *induces* a choice function $C^i$, under which $i$ chooses the subset of $Y$ that is highest-ranked according to the preference relation $\succ_i$; that is,

$$C^i(Y) = \max_{\succ_i}\{Z \subseteq X_i : Z \subseteq Y\},$$

where $\max_{\succ_i}$ indicates maximization with respect to the ordering $\succ_i$. Note that any choice function induced by a preference relation satisfies the irrelevance of rejected contracts condition (Aygün and Sönmez, 2013). For instance, the choice function induced by the preferences (1) of $h$ in our Sherlock–Watson example is represented by the following table:

| $Y$ | $C^h(Y)$ |
|---|---|
| $\{(s,h,r),(s,h,c),(w,h,c)\}$ | $\{(s,h,r),(w,h,c)\}$ |
| $\{(s,h,r),(w,h,c)\}$ | $\{(s,h,r),(w,h,c)\}$ |
| $\{(s,h,r),(s,h,c)\}$ | $\{(s,h,c)\}$ |
| $\{(s,h,c),(w,h,c)\}$ | $\{(s,h,c)\}$ |
| $\{(s,h,c)\}$ | $\{(s,h,c)\}$ |
| $\{(w,h,c)\}$ | $\{(w,h,c)\}$ |
| $\{(s,h,r)\}$ | $\{(s,h,r)\}$ |
| $\varnothing$ | $\varnothing$ |

.

Moreover, any preference relation that satisfies our key conditions—substitutability and the Law of Aggregate Demand—induces a choice function that satisfies the analogous conditions on choice functions.

**Proposition B.1.** *Any substitutable preference relation induces a substitutable choice function.*

*Proof.* Suppose that $\succ_i$ induces $C^i$ but that $C^i$ is not substitutable. As $C^i$ is not substitutable, there exists a set of contracts $Y$ and contracts $x$ and $z$ such that

- $z \notin C^i(Y \cup \{z\})$ but

- $z \in C^i(\{x\} \cup Y \cup \{z\})$.

By (3), the first fact means that $z$ must not be in the most-preferred subset of $Y \cup \{z\}$ under $\succ_i$, while the second fact implies that $z$ must be in the most-preferred subset of $\{x\} \cup Y \cup \{z\}$ under $\succ_i$. Thus, we see that $\succ_i$ is not substitutable. $\square$

**Proposition B.2.** *Any preference relation that satisfies the Law of Aggregate Demand induces a choice function that satisfies the Law of Aggregate Demand.*

*Proof.* Suppose that $\succ_i$ induces $C^i$ and that $\succ_i$ satisfies the Law of Aggregate Demand. For any sets of contracts $\hat{Y}$ and $Y$ such that $\hat{Y} \subseteq Y$, let $\hat{Z} = C^i(\hat{Y})$ and $Z = C^i(Y)$. Since $\hat{Z} = C^i(\hat{Y})$, we have that $\hat{Z}$ is the most-preferred set from $\hat{Y}$. Similarly, since $Z = C^i(Z)$, we have that $Z$ is the most-preferred set from $Y$. Thus, $|\hat{Z}| \leq |Z|$ as $\succ_i$ satisfies the Law of Aggregate Demand, and so $C^i$ satisfies the Law of Aggregate Demand by definition. $\square$

Finally, we show that our completion concept for preference relations links up to our completion concept for choice functions.

**Proposition B.3.** *If $\hat{\succ}_i$ is a completion of $\succ_i$, then the choice function $\hat{C}^i$ induced by $\hat{\succ}_i$ is a completion of the choice function $C^i$ induced by $\succ_i$.*

*Proof.* Suppose that $\hat{C}^i$ is not a completion of $C^i$. Then there exists a set of contracts $Y$ such that $\hat{Z} = \hat{C}^i(Y) \neq C^i(Y) = Z$ and $\hat{Z}$ is feasible; note that $Z$ is feasible as $C^i$ only chooses feasible sets. Moreover, since $\hat{C}^i$ is induced by $\hat{\succ}_i$, we have that $\hat{Z} \mathrel{\hat{\succ}_i} Z$, and, since $C^i$ is induced by $\succ_i$, we have that $Z \succ_i \hat{Z}$. But then $\hat{\succ}_i$ is not a completion of $\succ_i$, as $\hat{Z} \mathrel{\hat{\succ}_i} Z$, $Z \succ_i \hat{Z}$, and both $Z$ and $\hat{Z}$ are feasible. $\square$

## B.2 The Relationship between the Core, Stability, and Choice-Theoretic Stability

We now discuss the relationship between stability, choice-theoretic stability, and the strict core. For any preference profile (or extended preference profile) $\succ$, an outcome $A$ is in the *(strict) core* if there does not exist a set of agents $J$ and set of contracts $Z \subseteq \cup_{j \in J} X_j$ such that, $Z_j \succeq_j A_j$ for all agents $j \in J$ and $Z_j \succ_j A_j$ for some agent $j \in J$; we say that $(J, Z)$ *core-blocks* $A$.

**Proposition B.4.** *For any (extended) preference profile $\succ$, an outcome is in the strict core if and only if it is stable.*

*Proof.* Suppose $A$ is not stable. Then either $A$ is not individually rational or $A$ is blocked.

- If $A$ is not individually rational, then there exists an agent $i$ and a set $Z \subsetneq A_i$ such that $Z \succ_i A_i$ for some agent $i$. If $i$ is a doctor, then either $Z = \varnothing$ or $|A_i| > 1$; in either case, $(\{i\}, \varnothing)$ core-blocks $A$. If $i$ is a hospital, then $(\{i\} \cup \{d \in D : Z_d \neq \varnothing\}, Z)$ core-blocks $A$.

- If $A$ is blocked, then there exists a hospital $h$ and a set of contracts $Z$ and such that $Z \succ_h A$ and and $Z_d \succeq_d Y_d$ for all doctors $d$ such that $Z_d \neq \varnothing$. Thus, we have that $(\{d \in D : Z_d \neq \varnothing\} \cup \{h\}, Z)$ core-blocks $A$.

Suppose $A$ is not in the core. Then there exists $(J, Z)$ that core-blocks $A$. There are two cases:

- If $Z \subseteq A$, then $Z \subsetneq A$ as if $Z = A$ then no agent can strictly prefer $Z$ to $A$. Hence, $Z_j \subsetneq A_j$ for some agent $j$. Since $Z_j \neq A_j$ and $Z_j \succeq_j A_j$, we have that $Z_j \succ_j A_j$ (as preferences are strict). Hence, $A_j$ is not $j$'s most-preferred set from $A$ and so $A$ is not individually rational.

- If $Z \nsubseteq A$, then there exists $(d, h, t) \in Z \smallsetminus A$. Then $Z_h$ blocks $A$: $Z_h \succ_h A_h$ (as $(J, Z)$ core-blocks $A$ and $Z_h \neq A_h$) and $Z_d \succeq_d A_d$ (as $(J, Z)$ core-blocks $A$).

This completes the proof. $\square$

We now show that choice-theoretic stability is equivalent to stability, in the sense that an outcome $A$ is stable under a profile of preferences relations $\succ$ if and only if $A$ is choice-theoretic stable under the profile of choice functions induced by $\succ$.

**Proposition B.5.** *Let $C$ be the profile of choice functions induced by a preference relation $\succ$. Then $A$ is (choice-theoretic) stable under $C$ if and only if $A$ is stable under $\succ$.*

*Proof.* Suppose $A$ is not choice-theoretic stable under $C$. Then either $A$ is not individually rational under $C$ or $A$ is blocked under $C$:

- If $A$ is not individually rational under $C$, then there exists an agent $i$ such that $C^i(A) \subsetneq A_i$. Thus, $A_i$ is not $i$'s most-preferred set from $A_i$ under $\succ$, and so $A$ is not individually rational under $\succ$.

- If $A$ is blocked under $C$, then there exists a hospital $h$ and a nonempty set $\hat{Z} \subseteq X_h \smallsetminus A$ such that $\hat{Z}_i \subseteq C^i(A \cup \hat{Z})$ for all $i$ associated with contracts in $\hat{Z}$. Let $Z = C^h(A \cup \hat{Z})$. Since $\hat{Z}$ is nonempty and $\hat{Z} \subseteq C^h(A \cup \hat{Z}) = Z$, we have that $Z \succ_h A$.

  Also note that $Z = C^h(A \cup \hat{Z})$ contains at most one contract with each doctor since $h$ only chooses feasible sets of contracts under $C$ as $C$ is induced by a (non-extended) preference relation $\succ$. Thus, for a doctor $d$, if $Z_d \neq \varnothing$, there exists a unique contract $z \in Z$. There are two subcases:

  **$z \in A$:** In this case, $Z_d = A_d$ and so we have $Z_d \succeq_d A_d$.

  **$z \in \hat{Z}$:** In this case, since $\hat{Z}_d \subseteq C^d(A \cup \hat{Z})$, we have that $\{z\} \succ_d A_d$.

  Thus, $Z$ blocks $A$ under $\succ$.

If $A$ is (choice-theoretic) stable under $C$, the stability of $A$ under $\succ$ follows from Proposition B.6. □

However, we should note that the equivalence between stability and choice-theoretic stability of Proposition B.5 does not hold when we consider extended preference profiles. For instance, consider the extended preference relation (2) of our Sherlock–Watson example in Section 2, and suppose that Sherlock has preferences

$$\{(s,h,c)\} \succ_s \{(s,h,r)\} \succ_s \varnothing$$

while Watson preferences of

$$\{(w, h, c)\} \succ_w \varnothing.$$

There are two stable outcomes, $\{(s, h, c)\}$ and $\{(s, h, r), (w, h, c)\}$, while only $\{(s, h, c)\}$ is choice-theoretic stable. The outcome $\{(s, h, r), (w, h, c)\}$ is not choice-theoretic stable as $\{(s, h, c)\}$ is a block:

$$C^h(\{(s, h, c), (s, h, r), (w, h, c)\}) = \{(s, h, c), (s, h, r)\}$$

and

$$C^s(\{(s, h, c), (s, h, r), (w, h, c)\}) = \{(s, h, c)\}.$$

Note that the fact that $\{(s, h, c)\}$ is a block is subtle, as Sherlock and the hospital are not in agreement as to which set of contracts should be implemented through blocking. The hospital wants to hold onto the $(s, h, r)$ contract while adding the $(s, h, c)$ contract, while Sherlock wishes to switch from the $(s, h, r)$ contract to the $(s, h, c)$ contract.[42] Nevertheless, $\{(s, h, r), (w, h, c)\}$ is stable (in the sense of the main text), as $h$ only prefers $\{(s, h, c), (s, h, r)\}$ to $\{(s, h, r), (w, h, c)\}$, and $\{(s, h, c), (s, h, r)\}$ is not even individually rational for Sherlock.

Indeed, in the setting of many-to-many matching (with contracts), it is well-known that the core and the set of stable outcomes may both be non-empty and yet have an empty intersection (Blair, 1988). However, in our setting, any outcome that is stable under a completion $\hat{C}$ is also stable under the extended preference relation $\hat{\succsim}$ that induced $\hat{C}$.

**Proposition B.6.** *Let $\hat{C}$ be the profile of choice functions induced by an extended preference relation $\hat{\succsim}$ and suppose that $A$ is (choice-theoretic) stable under $\hat{C}$; then $A$ is stable under $\hat{\succsim}$.*

*Proof.* Suppose $A$ is not stable under $\hat{\succsim}$. Then either $A$ is not individually rational under $\hat{\succsim}$ or $A$ is blocked under $\hat{\succsim}$:

[42] This is a subtle issue in the definition of choice-theoretic stability that only arises in many-to-many matching with contracts (and more general models).

- If $A$ is not individually rational under $\hat{\succ}$, then there exists an agent $i$ such that $A_i$ is not $i$'s most-preferred set from $A_i$ under $\hat{\succ}$. Thus, $A_i \neq \hat{C}^i(A)$ and so $A$ is not individually rational under $\hat{C}$.

- If $A$ is blocked under $\hat{\succ}$ but individually rational, then there exists a a hospital $h$ and a set of contracts $Z$ such that $Z \hat{\succ}_h A$ and and $Z_d \hat{\succeq}_d Y_d$ for all doctors $d$ such that $Z_d \neq \varnothing$. Let $\hat{Z} = \hat{C}^h(Z \cup A) \smallsetminus A$. It is immediate that $\hat{Z} \subseteq X_h \smallsetminus A$ and that $\hat{Z} \subseteq \hat{C}^h(Z \cup A) = \hat{C}^h(\hat{Z} \cup A)$. Moreover, $\hat{Z}$ is nonempty as $\hat{C}^h(\hat{Z} \cup A) \hat{\succeq}_h Z \succ_h A_h$ and $A$ is individually rational for $h$, and so $A_h \hat{\succeq}_h W$ for all $W \subseteq A_h$.

  Moreover, for each doctor $d$ associated with a contract $z$ in $\hat{Z}$, we know that $\hat{Z}_d = \hat{C}^d(\hat{Z} \cup A)$ as doctors have unit demand and so $Z_d \hat{\succeq}_d Y_d$ if $Z_d \neq \varnothing$. Thus, $\hat{Z}$ is a block under $\hat{C}$.

This completes the proof. $\square$

### B.3 Deriving the Results Stated in the Main Text

Finally, we use our linkage results to prove the results stated in the main text:[43]

**Theorem 1:** Let $C$ be the profile of choice functions induced by $\succ$, and let $\hat{C}$ be the profile of choice functions induced by a substitutable completion $\hat{\succ}$ of $\succ$. Note that $\hat{C}$ is substitutable by Proposition B.1 and satisfies the irrelevance of rejected contracts condition as it is induced by a preference relation; moreover, $\hat{C}$ is a completion of $C$ by Proposition B.3. Hence, by Theorem A.1, there exists an outcome $A$ that is choice-theoretic stable with respect to $C$. Proposition B.5 then implies that $A$ that is stable with respect to $\succ$.

**Theorem 2:** Let $C$ be the profile of choice functions induced by $\succ$, and let $\hat{C}$ be the profile of choice functions induced by a substitutable completion $\hat{\succ}$ of $\succ$. Let $A$ be the outcome of the cumulative offer process under $\succ$; since (as we noted on page 31) the cumulative

[43]Note that Lemma 1 has already been proven in the main text.

offer process under $\succ$ proceeds identically to the cumulative offer process under $C$, the outcome of the cumulative offer process under $C$ must also be $A$. Theorem A.2 then implies that the outcome of the doctor-proposing cumulative offer process under $\hat{C}$ is $A$.[44] Finally, we have that the outcome of the cumulative offer process under $\hat{\succ}$ is also $A$, as the cumulative offer process under $\hat{\succ}$ proceeds identically to the cumulative offer process under $\hat{C}$.

Furthermore, since $\hat{C}$ satisfies the irrelevance of rejected contracts condition (as it is induced by a preference relation), Theorem A.2 implies that $A$ is choice-theoretic stable with respect to $C$. Proposition B.5 then implies that $A$ that is stable with respect to $\succ$.

**Theorem 3:** Let $C$ be the profile of choice functions induced by $\succ$, and let $\hat{C}$ be the profile of choice functions induced by a substitutable completion $\hat{\succ}$ of $\succ$ that satisfies the Law of Aggregate Demand. Note that $\hat{C}$ is substitutable by Proposition B.1, satisfies the Law of Aggregate Demand by Proposition B.2, satisfies the irrelevance of rejected contracts condition (as it is induced by a preference relation), and is a completion of $C$ by Proposition B.3. Thus, since the cumulative offer process under $\succ$ proceeds identically to the cumulative offer process under $C$, Theorem A.3 implies that the cumulative offer process is strategy-proof.

# C The Doctor-Proposing Deferred Acceptance Algorithm

The (doctor-proposing) deferred acceptance process under $\succ$ proceeds as follows.

**Step 1:** Each doctor *proposes* his most-preferred contract from $X$ under $\succ$ (assuming there is one); the set of proposed contracts is denoted $R^1$. Each hospital $h$ *holds* its favorite set of contracts from those that have been proposed; we call the set of held contracts $Y^1$.

**Step $\tau$:** Each doctor not associated with a currently held contract, i.e., without a contract

[44] Note that $\hat{C}$ is substitutable by Proposition B.1 and is a completion of $C$ by Proposition B.3.

in $Y^{\tau-1}$ proposes his most-preferred contract that has not yet been proposed (if any), i.e., his most preferred contract from $X \smallsetminus \cup_{\sigma=1}^{\tau-1} R^{\sigma}$. If no contract is proposed, then the algorithm terminates and the outcome is the set of contracts held by the hospitals, $Y^{\tau-1}$. Otherwise, the set of contracts proposed in Step $\tau$ is denoted $R^{\tau}$; each hospital $h$ holds its most-preferred set of contracts $Y_h^{\tau}$ from $R_h^{\tau} \cup Y_h^{\tau-1}$, i.e., those that have been proposed this period to $h$ and those currently held by $h$; we then take $Y^{\tau} = \cup_{h \in H} Y_h^{\tau}$ The algorithm then proceeds to Step $\tau + 1$.

# D Bilaterally Substitutable Preferences

Hatfield and Kojima (2010) introduced the bilateral substitutability condition, which is weaker than substitutability but nevertheless sufficient to guarantee the existence of stable many-to-one matching with contracts outcomes. Here, we show that there exist substitutably completable choice functions that are not bilaterally substitutable.

First, we recall the formal statement of the bilateral substitutability condition: We denote by $\mathsf{d}(x)$ the doctor associated with contract $x$ and by $\mathsf{h}(x)$ the hospital associated with contract $x$.

**Definition 3.** The choice function $C^i$ of $i \in D \cup H$ is *bilaterally substitutable* if for all $x, z \in X$ and $Y \subseteq X$ such that $\mathsf{d}(x), \mathsf{d}(z) \notin \mathsf{d}(Y)$, if $z \notin C^i(Y \cup \{z\})$, then $z \notin C^i(\{x\} \cup Y \cup \{z\})$.

The choice function induced by the preferences (1) of the hospital in our Sherlock–Watson example are in fact bilaterally substitutable. However, there are substitutably completable choice functions that are not bilaterally substitutable. For example, let $H = \{h\}$, $D = \{d, e, f\}$, and $X = \{x, y, \hat{y}, z\}$ where $h = \mathsf{h}(x) = \mathsf{h}(y) = \mathsf{h}(\hat{y}) = \mathsf{h}(z)$, $d = \mathsf{d}(x)$, $e = \mathsf{d}(y) = \mathsf{d}(\hat{y})$, and $f = \mathsf{d}(z)$. Consider the hospital preference relation

$$\{x, y, z\} \succ_h \{\hat{y}\} \succ_h \{x, y\} \succ_h \{x, z\} \succ_h \{y, z\} \succ_h \{y\} \succ_h \{x\} \succ_h \{z\} \succ \varnothing.$$

The preference relation $\succ_h$ induces a choice function $C^h$ that is not bilaterally substitutable.[45]

[45] Note that $z \notin \{\hat{y}\} = C^h(\{y, \hat{y}, z\})$, but $z \in \{x, y, z\} = C^h(\{x, y, \hat{y}, z\})$, even though $\mathsf{d}(x), \mathsf{d}(z) \notin \mathsf{d}(\{y, \hat{y}\})$.

Even though $C^h$ is not bilaterally substitutable, it may be substitutably completed: the choice function induced by the extended preference relation

$$\boldsymbol{\{y, \hat{y}\}} \hat{\succ}_h \{x, y, z\} \hat{\succ}_h \{\hat{y}\} \hat{\succ}_h \{x, y\} \hat{\succ}_h \{x, z\} \hat{\succ}_h \{y, z\} \hat{\succ}_h \{y\} \hat{\succ}_h \{x\} \hat{\succ}_h \{z\} \hat{\succ}_h \varnothing$$

is substitutable, satisfies the irrelevance of rejected contracts condition, and completes $C^h$.

The preceding example demonstrates that bilateral substitutability does not imply substitutable completability. Thus, we see that substitutable completability is truly a "new" sufficient condition for the existence of stable outcomes in the setting of many-to-one matching with contracts—it includes a class of choice functions that were not previously known to have stable outcomes guaranteed.

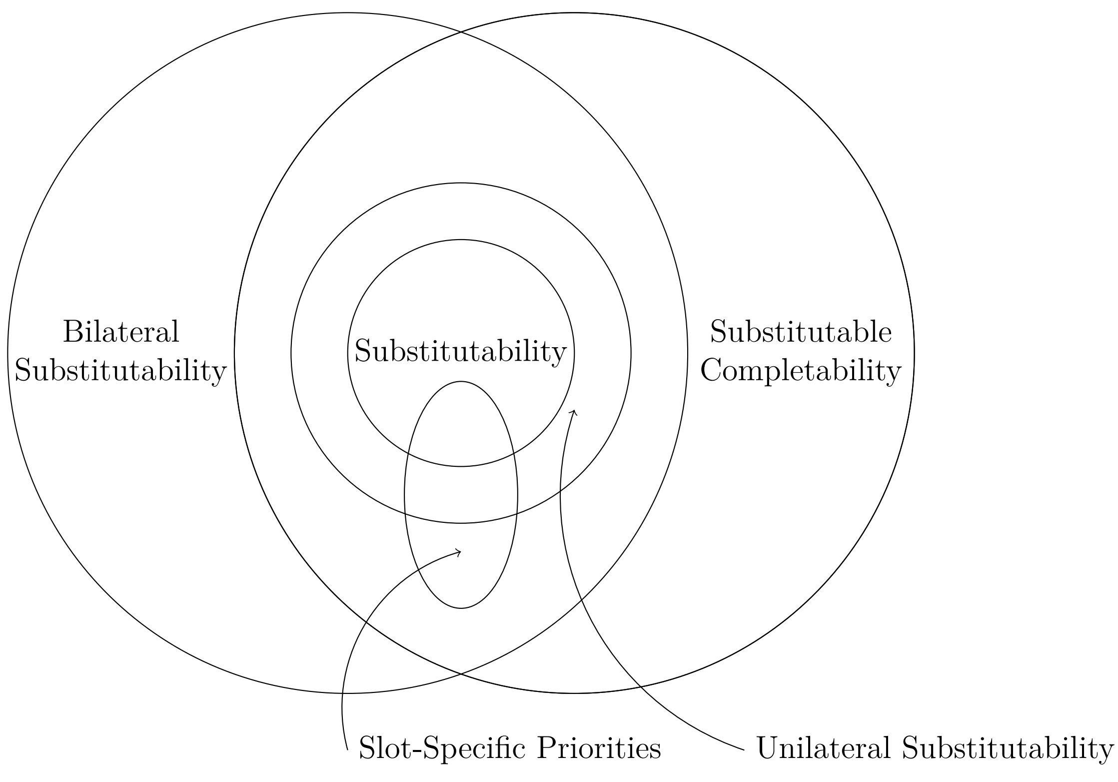


Figure 1: The relationship between substitutability concepts for many-to-one matching with contracts.

Interestingly, however, substitutable completability is not strictly weaker than bilateral substitutability. To see this, we consider a setting where $D = \{d, e\}$, $H = \{h\}$, and

$X = \{x, y, \hat{x}, \hat{y}\}$, with $h = \mathsf{h}(x) = \mathsf{h}(\hat{x}) = \mathsf{h}(y) = \mathsf{h}(\hat{y})$, $d = \mathsf{d}(x) = \mathsf{d}(\hat{x})$, and $e = \mathsf{d}(y) = \mathsf{d}(\hat{y})$. Consider the choice function $C^h$ induced by the preference relation

$$\{x, y\} \succ_h \{\hat{x}\} \succ_h \{\hat{y}\} \succ_h \{x\} \succ_h \{y\} \succ_h \varnothing.$$

It is straightforward to check that $C^h$ is bilaterally substitutable. But suppose that there were a substitutable completion $\hat{C}^h$ of $C^h$: We would need to have $\hat{C}^h(\{\hat{x}, y\}) = \{\hat{x}\}$ and $\hat{C}^h(\{\hat{x}, \hat{y}\}) = \{\hat{x}\}$, as $\hat{C}^h$ completes $C^h$; these facts imply that

$$\hat{C}^h(\{\hat{x}, y, \hat{y}\}) = \{\hat{x}\}, \tag{6}$$

as $\hat{C}^h$ is substitutable. As $\hat{C}^h$ completes $C^h$, we would also need to have $\hat{C}^h(\{x, \hat{y}\}) = \{\hat{y}\}$; this fact, along with (6), would imply that

$$\hat{C}^h(\{x, \hat{x}, y, \hat{y}\}) = \{\hat{x}\},$$

as $\hat{C}^h$ is substitutable. But then $\hat{C}^h$ could not be a completion—a contradiction—as $C^h(\{x, \hat{x}, y, \hat{y}\}) = \{x, y\} \neq \{\hat{x}\} = \hat{C}^h(\{x, \hat{x}, y, \hat{y}\})$ and $\hat{C}^h(\{x, \hat{x}, y, \hat{y}\}) = \{\hat{x}\}$ does not contain two contracts with the same doctor.

Figure 1 shows the relationship between substitutable completability and the substitutability structures introduced in this prior literature (assuming the irrelevance of rejected contracts condition).

# E Tasks-and-Slots Priorities

In this appendix, we describe a class of *tasks-and-slots priorities* that generalizes the *slot-specific priorities* of Kominers and Sönmez (2016).

As in Appendix D, we denote by $\mathsf{d}(x)$ the doctor associated with contract $x$; similarly, we denote by $\mathsf{d}(Y)$ the set of doctors associated with some contract in $Y$, i.e., $\mathsf{d}(Y) = \cup_{y \in Y} \mathsf{d}(y)$.

For each hospital $h$, there is a set of *slots* $\mathcal{S}^h$ and a (disjoint) set of *tasks* $\mathcal{T}^h$; the set of *positions* $\mathcal{P}^h$ is the union of slots and tasks, i.e., $\mathcal{P}^h \equiv \mathcal{S}^h \cup \mathcal{T}^h$. For each slot $s \in \mathcal{S}^h$, there exists a *preference ordering* $\succ_s$ over elements of $X$ and an *outside option* $\emptyset$.

Similarly, for each task $t \in \mathcal{T}^h$, there exists a *preference ordering* $\succ_t$ over elements of $X$ and an *outside option* $\emptyset$. However, the set of tasks can be partitioned into a set of classes $\mathscr{C}$ where, for any two tasks $t, \bar{t}$ in the same class $\mathcal{C} \in \mathscr{C}$, the tasks have identical preference orderings, i.e., $\succ_t = \succ_{\bar{t}}$, while tasks in distinct classes find disjoint sets of contracts acceptable.[46] Finally, each hospital $h$ is also endowed with a *precedence ordering* $\rhd_h^Y$ over positions in $\mathcal{P}^h$ that determines, as a function of the set of proposed contracts $Y$, the order in which positions will be filled.

We impose the following restrictions on the precedence ordering $\rhd_h$:

1. Tasks are filled before slots; that is, for all $Y \subseteq X$, for any task $t \in \mathcal{T}^h$ and any slot $s \in \mathcal{S}^h$, we have that $t \rhd_h^Y s$.

2. Slots are filled in the same order regardless of the set of contracts available; that is, for all $Y, \bar{Y} \subseteq X$, for any slots $s, \bar{s} \in \mathcal{S}^h$, if $s \rhd_h^Y \bar{s}$ then $s \rhd_h^{\bar{Y}} \bar{s}$.

Finally, the hospital has a *quota* $q^h$ of positions it wishes to fill; we assume that $q^h \geq |\mathcal{T}^h|$. A *tasks-and-slots preference structure* is a tuple $(\mathcal{T}^h, \mathscr{C}, \mathcal{S}^h, \{\succ_t\}_{t \in \mathcal{T}^h}, \{\succ_s\}_{s \in \mathcal{S}^h}, \rhd_h, q^h)$.

If the choice function $C^h$ of hospital $h$ is *induced by the tasks-and-slots preference structure* $(\mathcal{T}^h, \mathscr{C}, \mathcal{S}^h, \{\succ_t\}_{t \in \mathcal{T}^h}, \{\succ_s\}_{s \in \mathcal{S}^h}, \rhd_h, q^h)$, then we compute $C^h(Y)$ for any set of available contracts $Y$ as follows:

1. Initialize the set of *available contracts* as $A^0 = Y$ and the set of *selected contracts* as $G^0 = \varnothing$.

2. Label the positions in $\mathcal{P}^h$ as $p^1, p^2, \ldots, p^{|\mathcal{P}^h|}$, where $p^\ell$ is the $\ell^{\text{th}}$ highest position according to the precedence order $\rhd_h^Y$.

3. If the number of held contracts is equal to the quota, i.e., $|G^{\ell-1}| = q^h$, or if all the positions have been considered, i.e., $\ell = |\mathcal{P}^h| + 1$, continue to Step 4. Otherwise, let $x^\ell$ be the $\succ_{p^\ell}$-maximal contract in $A^{\ell-1} \cup \{\emptyset\}$. If $x^\ell \neq \emptyset$, then:

[46] We say that a given contract $x$ is *acceptable* for a given position $p$ if it is preferred to the null contract, i.e., $x \succ_p \emptyset$.

(a) add $x^\ell$ to the set of selected contracts, i.e., let $G^\ell \equiv G^{\ell-1} \cup \{x^\ell\}$; and

(b) remove any contracts associated with $\mathsf{d}(x^\ell)$ from the set of available contracts, i.e., let $A^\ell \equiv A^{\ell-1} \smallsetminus Y_{\mathsf{d}(x^\ell)}$.

If instead $x^\ell = \emptyset$, let $G^\ell = G^{\ell-1}$ and $A^\ell = A^{\ell-1}$. Increment $\ell$ and return to Step 3.

4. Finally, take the choice of $h$ from $Y$ to be the set of selected contracts, i.e., set $C^h(Y) = G^{\ell-1}$.

If the contract $x$ is added to $G^\ell$ in Step $\ell$, then we say that $z$ *fills position* $\mathfrak{p}^\ell$ *according to the precedence order* $\rhd_h^Y$.

As constructed, a choice function induced by a tasks-and-slots preference structure is not necessarily substitutable nor does it necessarily satisfy the Law of Aggregate Demand. In fact, it may not necessarily satisfy the irrelevance of rejected contracts condition, as we demonstrate in Appendix E.1.

We now show the main result of this appendix.

**Theorem E.1.** *Any choice function induced by a tasks-and-slots preference structure has a substitutable completion that satisfies the Law of Aggregate Demand and the irrelevance of rejected contracts condition.*

We suppose that the choice function $C^h$ is induced by the tasks-and-slots preference structure $(\mathcal{T}^h, \mathscr{C}, \mathcal{S}^h, \{\succ_t\}_{t\in\mathcal{T}^h}, \{\succ_s\}_{s\in\mathcal{S}^h}, \rhd_h, q^h)$. We construct a completion $\hat{C}^h$ of $C^h$ by relaxing the constraint that the hospital can choose at most one contract with each doctor. That is, under $\hat{C}^h$, when a contract $x$ is chosen, we remove only the contract $x$ from consideration for other positions, instead of removing all the contracts with the doctor $\mathsf{d}(x)$. More formally, $\hat{C}^h$ is the *completion induced by the tasks-and-slots preference structure* $(\mathcal{T}^h, \mathscr{C}, \mathcal{S}^h, \{\succ_t\}_{t\in\mathcal{T}^h}, \{\succ_s\}_{s\in\mathcal{S}^h}, \rhd_h, q^h)$, and is generated by the following algorithm:

1. Initialize the set of *available contracts* as $\hat{A}^0 = Y$ and the set of *selected contracts* as $\hat{G}^0 = \varnothing$.

2. Label the positions in $\mathcal{P}^h$ as $p^1, p^2, \ldots, p^{|\mathcal{P}^h|}$, where $p^\ell$ is the $\ell^{\text{th}}$ highest position according to the precedence order $\rhd_h^Y$.

3. If the number of held contracts is equal to the quota, i.e., $|\hat{G}^{\ell-1}| = q^h$, or if all the positions have been considered, i.e., $\ell = |\mathcal{P}^h| + 1$, continue to Step 4. Otherwise, let $\hat{x}^\ell$ be the $\succ_{p^\ell}$-maximal contract in $\hat{A}^{\ell-1} \cup \{\emptyset\}$. If $\hat{x}^\ell \neq \emptyset$, then:

   (a) add $\hat{x}^\ell$ to the set of selected contracts, i.e., let $\hat{G}^\ell \equiv \hat{G}^{\ell-1} \cup \{\hat{x}^\ell\}$; and

   (b) remove $\hat{x}^\ell$ from the set of available contracts, i.e., let $\hat{A}^\ell \equiv \hat{A}^{\ell-1} \smallsetminus \{\hat{x}^\ell\}$.

   If instead $\hat{x}^\ell = \emptyset$, let $\hat{G}^\ell = \hat{G}^{\ell-1}$ and $\hat{A}^\ell = \hat{A}^{\ell-1}$. Increment $\ell$ and return to Step 3.

4. Finally, take the choice of $h$ from $Y$ to be the set of selected contracts, i.e., set $C^h(Y) = \hat{G}^{\ell-1}$.

Note that $\hat{C}^h$ is defined using the same algorithm as $C^h$ except that in Step 3b of the computation of $\hat{C}^h(Y)$, we remove just $\{\hat{x}^\ell\}$ from consideration for lower-precedence positions, while in Step 3b of the computation of $C^h(Y)$, we remove $Y_{\mathsf{d}(x^\ell)} \supseteq \{x^\ell\}$ from consideration for lower-precedence positions.

**Claim 1.** *The choice function $\hat{C}^h$ completes $C^h$.*

*Proof.* It suffices to show that for each $Y \subseteq X$, if $\hat{C}^h(Y) \neq C^h(Y)$, then there is some doctor $d \in D$ such that $\hat{C}^h(Y)$ contains two contracts associated with $d$.

If $\hat{C}^h(Y) \neq C^h(Y)$, then there is some first instance for which $x^\ell \neq \hat{x}^\ell$, i.e., some minimal $\ell$ such that $x^\ell \neq \hat{x}^\ell$. Now, the only difference between the algorithm defining $C^h$ and that defining $\hat{C}^h$ arises in Step 3b: in computing $C^h(Y)$, for each $m < \ell$, we set $A^m = A^{m-1} \smallsetminus Y_{\mathsf{d}(x^m)}$, whereas in computing $\hat{C}^h(Y)$, we set $\hat{A}^m = \hat{A}^{m-1} \smallsetminus \{\hat{x}^m\}$. Thus, since $x^m = \hat{x}^m$ for all $m \leq \ell$ by construction, we see that $A^{\ell-1}$, the set of contracts available to be assigned to $h$ in iteration $\ell$ of Step 3 of the computation of $C^h(Y)$, differs from $\hat{A}^{\ell-1}$ (the set of contracts available to be assigned to $h$ in iteration $\ell$ of Step 3 of the computation of

$\hat{C}^h(Y)$) only in that additional contracts with doctors in $\mathsf{d}(G^{\ell-1})$ are available; specifically, $\hat{A}^{\ell-1} = A^{\ell-1} \cup (Y_{\mathsf{d}(G^{\ell-1})} \smallsetminus G^{\ell-1})$.

Now, the contract $\hat{x}^\ell$, selected in iteration $\ell$ of Step 3 of the computation of $\hat{C}^h(Y)$, differs from $x^\ell$, the contract selected in iteration $\ell$ of Step 3 of the computation of $C^h(Y)$. Moreover, $\hat{x}^\ell$ is maximal among contracts in the set $\hat{A}^{\ell-1}$ of contracts available to be assigned in iteration $\ell$ of the computation of $\hat{C}^h(Y)$. Thus, we have that $\hat{x}^\ell \in \hat{A}^{\ell-1} \smallsetminus A^{\ell-1} = Y_{\mathsf{d}(G^{\ell-1})} \smallsetminus G^{\ell-1}$; so, in particular, $\mathsf{d}(\hat{x}^\ell) \in \mathsf{d}(G^{\ell-1})$. Hence, when computing $\hat{C}^h(Y)$, we have that $\hat{G}^m$ contains at least two contracts associated with the doctor $\mathsf{d}(\hat{x}^\ell)$ for all $m \geq \ell$. Hence, $\hat{C}^h(Y)$ contains at least two contracts associated with the doctor $\mathsf{d}(\hat{x}^\ell)$. $\square$

**Claim 2.** *The completion $\hat{C}^h$ induced by the tasks-and-slots preference structure*

$$(\mathcal{T}^h, \mathscr{C}, \mathcal{S}^h, \{\succ_t\}_{t\in\mathcal{T}^h}, \{\succ_s\}_{s\in\mathcal{S}^h}, \rhd_h, q^h)$$

*is equivalent to the completion induced by the tasks-and-slots preference structure*

$$(\mathcal{T}^h, \mathscr{C}, \mathcal{S}^h, \{\succ_t\}_{t\in\mathcal{T}^h}, \{\succ_s\}_{s\in\mathcal{S}^h}, \blacktriangleright_h, q^h),$$

*where $\blacktriangleright_h^Y = \rhd_h^\varnothing$ for all $Y \subseteq X$.*

*Proof.* Let $m$ denote the total number of tasks, i.e., $m = |\mathcal{T}^h|$. Recall that all tasks in a given class $\mathcal{C}$ use the same preference ordering; we abuse notation slightly by denoting that preference ordering $\succ_{\mathcal{C}}$. Let

$$M_{\mathcal{C}} \equiv \{x \in Y : x \succ_{\mathcal{C}} \emptyset \text{ and } x \text{ is one of the } |\mathcal{C}| \text{ highest-ranked elements of } Y \text{ according to } \succ_{\mathcal{C}}\}.$$

Now, *for any precedence order*, as any two tasks in different classes find disjoint sets of contracts acceptable, and any two tasks in the same class agree on the preference ordering over contracts, we compute that $\hat{G}^m = \cup_{\mathcal{C}\in\mathscr{C}} M_{\mathcal{C}}$, as the quota $q^h$ is at least $|\mathcal{T}^h|$. It then follows that, again for any precedence ordering, the set of available contracts at the end of iteration $m$ of the computation $\hat{C}^h(Y)$ is exactly $Y \smallsetminus \hat{G}^m$.

Moreover, for every precedence order, slots are filled only after tasks are considered, and slots are always filled in the same order. Hence, as for any precedence order the set of contracts available to be assigned to slots is always $Y \smallsetminus \hat{G}^m$, the set of contracts assigned to slots $(\hat{G}^{q^h} \smallsetminus \hat{G}^m)$ is independent of the precedence order.

It follows that the set of contracts chosen by the completion $\hat{C}^h$ induced by the tasks-and-slots preference structure $(\mathcal{T}^h, \mathscr{C}, \mathcal{S}^h, \{\succ_t\}_{t \in \mathcal{T}^h}, \{\succ_s\}_{s \in \mathcal{S}^h}, \rhd_h, q^h)$ is the same as the set of contracts chosen by the completion induced by the tasks-and-slots preference structure $(\mathcal{T}^h, \mathscr{C}, \mathcal{S}^h, \{\succ_t\}_{t \in \mathcal{T}^h}, \{\succ_s\}_{s \in \mathcal{S}^h}, \blacktriangleright_h, q^h)$. $\square$

Given Claim 2, it is without loss of generality to assume that $\rhd_h^Y = \rhd_h^\varnothing$ for all $Y \subseteq X$, i.e., that $\rhd_h$ is a fixed precedence order. Accordingly, we shall drop the superscript on $\rhd_h$ for the remainder of the proof.

**Claim 3.** *The completion $\hat{C}^h$ induced by the tasks-and-slots preference structure*

$$(\mathcal{T}^h, \mathscr{C}, \mathcal{S}^h, \{\succ_t\}_{t \in \mathcal{T}^h}, \{\succ_s\}_{s \in \mathcal{S}^h}, \rhd_h, \bar{q}^h)$$

*is substitutable and satisfies the Law of Aggregate Demand.*

*Proof.* For any set of contracts $Y$, we let $\hat{A}_Y^\ell$ denote the set of contracts available to be assigned to positions after iteration $\ell$ of Step 3 of the computation of $\hat{C}^h(Y)$. Analogously, we let $\hat{G}_Y^\ell$ denote the set of contracts selected by the end of iteration $\ell$ of Step 3 of the computation of $\hat{C}^h(Y)$.

To show that $\hat{C}^h$ is substitutable, we show that for any $z \in X_h$ and $Y \subseteq \hat{Y} \subseteq X_h$,

$$\text{if } z \notin \hat{C}^h(Y) \text{ but } z \in Y, \text{ then } z \notin \hat{C}^h(\hat{Y}). \tag{7}$$

To show that $\hat{C}^h$ satisfies the Law of Aggregate Demand, we show that for any $Y \subseteq \hat{Y} \subseteq X_h$, we have that

$$|\hat{C}(Y)| \leq |\hat{C}(\hat{Y})|. \tag{8}$$

We show both (7) and (8) show by way of the following claim:

**Subclaim 1.** *At each iteration $\ell$ of Step 3 of the computations of $\hat{C}^h(Y)$ and $\hat{C}^h(\hat{Y})$, we have that $\hat{A}_Y^\ell \subseteq \hat{A}_{\hat{Y}}^\ell$ and $|\hat{G}_Y^\ell| \leq |\hat{G}_{\hat{Y}}^\ell|$.*

*Proof.* We proceed by induction. First, we note that for $\ell = 0$, we have $\hat{A}^0_{\hat{Y}} = \hat{Y} \supseteq Y = \hat{A}^0_Y$ and $\hat{G}^0_{\hat{Y}} = \varnothing = \hat{G}^0_Y$, so we assume that the claim holds for all $m < \ell$. At iteration $\ell > 0$, let $p$ be the $\ell^{\text{th}}$ highest position according to the precedence ordering $\rhd_h$, and let $x^\ell_Y$ be the $\succ_p$-maximal contract in $\hat{A}^\ell_Y$ and $x^\ell_{\hat{Y}}$ be the $\succ_p$-maximal contract in $\hat{A}^\ell_{\hat{Y}}$. There are four possibilities:

**Case 1: $x^\ell_{\hat{Y}} = x^\ell_Y \neq \emptyset$.** In this case, $\hat{A}^\ell_{\hat{Y}} = \hat{A}^{\ell-1}_{\hat{Y}} \smallsetminus \{x^\ell_Y\}$ and $\hat{A}^{\ell-1}_Y = \hat{A}^\ell_Y \smallsetminus \{x^\ell_Y\}$. Since by the inductive hypothesis we have $\hat{A}^{\ell-1}_{\hat{Y}} \supseteq \hat{A}^{\ell-1}_Y$, it immediately follows that $\hat{A}^\ell_{\hat{Y}} \supseteq \hat{A}^\ell_Y$. Moreover, since by the inductive hypothesis we have $|\hat{G}^{\ell-1}_{\hat{Y}}| \geq |\hat{G}^{\ell-1}_Y|$, we know that $|\hat{G}^\ell_{\hat{Y}}| = |\hat{G}^{\ell-1}_{\hat{Y}}| + 1 \geq |\hat{G}^{\ell-1}_Y| + 1 = |\hat{G}^\ell_Y|$.

**Case 2: $x^\ell_{\hat{Y}} = x^\ell_Y = \emptyset$.** In this case, $\hat{A}^\ell_{\hat{Y}} = \hat{A}^{\ell-1}_{\hat{Y}}$ and $\hat{A}^\ell_Y = \hat{A}^{\ell-1}_Y$; moreover, $\hat{G}^\ell_{\hat{Y}} = \hat{G}^{\ell-1}_{\hat{Y}}$ and $\hat{G}^\ell_Y = \hat{G}^{\ell-1}_Y$. As by the inductive hypothesis we have $\hat{A}^{\ell-1}_{\hat{Y}} \supseteq \hat{A}^{\ell-1}_Y$, it immediately follows that $\hat{A}^\ell_{\hat{Y}} \supseteq \hat{A}^\ell_Y$. Moreover, since by the inductive hypothesis we have $|\hat{G}^{\ell-1}_{\hat{Y}}| \geq |\hat{G}^{\ell-1}_Y|$, we know that $|\hat{G}^\ell_{\hat{Y}}| = |\hat{G}^{\ell-1}_{\hat{Y}}| \geq |\hat{G}^{\ell-1}_Y| = |\hat{G}^\ell_Y|$.

**Case 3: $x^\ell_{\hat{Y}} \neq x^\ell_Y$ and $x^\ell_Y = \emptyset$.** In this case, note that $x^\ell_{\hat{Y}} \neq \emptyset$ implies that $x^\ell_{\hat{Y}} \succ_p \emptyset$. This implies that $x^\ell_{\hat{Y}} \notin \hat{A}^{\ell-1}_Y$, as otherwise we would not have $x^\ell_Y = \emptyset$. Since by the inductive hypothesis we have $\hat{A}^{\ell-1}_{\hat{Y}} \supseteq \hat{A}^{\ell-1}_Y$, it immediately follows that $\hat{A}^\ell_{\hat{Y}} = \hat{A}^{\ell-1}_{\hat{Y}} \smallsetminus \{x^\ell_{\hat{Y}}\} \supseteq \hat{A}^{\ell-1}_Y = \hat{A}^\ell_Y$. Moreover, since by the inductive hypothesis we have $|\hat{G}^{\ell-1}_{\hat{Y}}| \geq |\hat{G}^{\ell-1}_Y|$, we know that $|\hat{G}^\ell_{\hat{Y}}| = |\hat{G}^{\ell-1}_{\hat{Y}}| + 1 \geq |\hat{G}^{\ell-1}_Y| = |\hat{G}^\ell_Y|$.

**Case 4: $x^\ell_{\hat{Y}} \neq x^\ell_Y$ and $x^\ell_Y \neq \emptyset$.** First, we note that $x^\ell_{\hat{Y}} \succ_p x^\ell_Y$, as by the inductive hypothesis we have $\hat{A}^{\ell-1}_{\hat{Y}} \supseteq \hat{A}^{\ell-1}_Y$ and $p^\ell$ is assigned the $\succ_p$-maximal contract in Step 3.[47] Hence, we must have $x^\ell_{\hat{Y}} \notin \hat{A}^{\ell-1}_Y$, as otherwise $x^\ell_Y \neq x^\ell_{\hat{Y}}$ would not be selected in the $\ell^{\text{th}}$ iteration of Step 3 of the compuation of $\hat{C}^h(Y)$. Since by the inductive hypothesis we have $\hat{A}^{\ell-1}_{\hat{Y}} \supseteq \hat{A}^{\ell-1}_Y$, it immediately follows that $\hat{A}^\ell_{\hat{Y}} = \hat{A}^{\ell-1}_{\hat{Y}} \smallsetminus \{x^\ell_{\hat{Y}}\} \supseteq \hat{A}^{\ell-1}_Y \smallsetminus \{x^\ell_Y\} = \hat{A}^\ell_Y$. Moreover, since by the inductive hypothesis we have $|\hat{G}^{\ell-1}_{\hat{Y}}| \geq |\hat{G}^{\ell-1}_Y|$, we know that $|\hat{G}^\ell_{\hat{Y}}| = |\hat{G}^{\ell-1}_{\hat{Y}}| + 1 \geq |\hat{G}^{\ell-1}_Y| + 1 = |\hat{G}^\ell_Y|$. $\square$

[47] In particular, this implies that $x^\ell_{\hat{Y}} \neq \emptyset$.

Subclaim 1 implies the substitutability of $\hat{C}^h$ (that is, (7)), as: For each iteration $\ell$ of Step 3, the $\ell^{\text{th}}$ highest-precedence position $p^\ell$ is assigned the $\succ_{p^\ell}$-maximal contract from the set of contracts still available. Thus, if $z \notin \hat{C}^h(Y)$, then $z$ is not selected in any iteration of Step 3 the computation of $\hat{C}^h(Y)$, so it must be that $z$ is not the $\succ_{p^\ell}$-maximal element of $\hat{A}_Y^{\ell-1} \cup \{\emptyset\}$ for any $\ell$ reached in the computation of $\hat{C}^h(Y)$. But then, as $\hat{A}_{\hat{Y}}^{\ell-1} \cup \{\emptyset\} \supseteq \hat{A}_Y^{\ell-1} \cup \{\emptyset\}$ (by Claim 1), we see that $z$ can not be the $\succ_{p^\ell}$-maximal element of $\hat{A}_{\hat{Y}}^{\ell-1} \cup \{\emptyset\}$ for any $\ell$ reached in the computation of $\hat{C}^h(Y)$. Moreover, we have (again by Claim 1) that $|\hat{G}_{\hat{Y}}^\ell| \geq |\hat{G}_Y^\ell|$; hence if the computation of $\hat{C}^h(Y)$ stops at iteration $\ell$ of Step 3, then the computation of $\hat{C}^h(\hat{Y})$ must stop at iteration $\hat{\ell} \leq \ell$. Thus, we see that $z$ can not be selected in the computation of $\hat{C}^h(\hat{Y})$.

Subclaim 1 also implies that $\hat{C}^h$ satisfies the Law of Aggregate Demand (that is, (8)), as: For each iteration $\ell$ of Step 3, we have that $|\hat{G}_{\hat{Y}}^\ell| \geq |\hat{G}_Y^\ell|$, so at any iteration $\ell$ of Step 3 before the quota is met, more contracts are assigned in the computation of $\hat{C}^h(\hat{Y})$ than in the computation of $\hat{C}^h(Y)$. Thus, if the computation of $\hat{C}^h(\hat{Y})$ ends at iteration $|\mathcal{P}^h| + 1$ of Step 3 (and, hence, the computation of $\hat{C}^h(Y)$ also ends at iteration $|\mathcal{P}^h| + 1$), we have that $|\hat{C}^h(\hat{Y})| = |\hat{G}_{\hat{Y}}^{|\mathcal{P}^h|}| \geq |\hat{G}_Y^{|\mathcal{P}^h|}| = |\hat{C}^h(Y)|$. Moreover, the computation of $\hat{C}^h(\hat{Y})$ ends at iteration $\ell < |\mathcal{P}^h| + 1$ of Step 3 only if $|\hat{G}_{\hat{Y}}^{\ell-1}| = q^h$. But in this case, the result is immediate, as $|\hat{C}^h(W)| \leq q^h = |\hat{G}_{\hat{Y}}^{\ell-1}| = \hat{C}^h(\hat{Y})$ for all $W \subseteq X$ (and, in particular, when $W = Y$). $\square$

**Claim 4.** *The completion $\hat{C}^h$ induced by the tasks-and-slots preference structure*

$$(\mathcal{T}^h, \mathscr{C}, \mathcal{S}^h, \{\succ_t\}_{t\in\mathcal{T}^h}, \{\succ_s\}_{s\in\mathcal{S}^h}, \rhd_h, \bar{q}^h)$$

*satisfies the irrelevance of rejected contracts condition.*

*Proof.* Proposition 1 of Aygün and Sönmez (2012) shows that any substitutable choice function that satisfies the Law of Aggregate Demand also satisfies the irrelevance of rejected contracts condition.[48] Thus, our claim here follows directly from Claim 3. $\square$

Taken together, the claims of this section show Theorem E.1.

[48] Although Aygün and Sönmez consider a many-to-one matching with contracts setting, their proof extends without change to the many-to-many matching with contracts setting.

## E.1 A Choice Function Induced by a Tasks-and-Slots Preference Structure That Does Not Satisfy the Irrelevance of Rejected Contracts Condition

Under a tasks-and-slots preference structure, precedence orders can depend arbitrarily on the set of contracts available: in particular, they can depend on contracts which are unacceptable. Thus, the irrelevance of rejected contracts condition can naturally be violated, as the presence of an unacceptable contract can change the precedence order in a way that changes the set of contracts chosen.

For a simple example, let $D = \{d, e\}$, $H = \{h\}$, and $T = \{c, r\}$, where the contractual term $c$ denotes working as a clinician and the contractual term $r$ denotes working as a researcher. The set of contracts is given by $X = D \times \{h\} \times \{c, r\}$.

Hospital $h$ has two positions, a clinician task and a researcher task, denoted $\mathcal{T}^h = \{\mathfrak{c}, \mathfrak{r}\}$; the set of slots $\mathcal{S}^h$ is empty. The preference orderings for the tasks are:

$$\succ_{\mathfrak{c}} : (d, h, c) \succ \emptyset$$

$$\succ_{\mathfrak{r}} : (d, h, r) \succ \emptyset$$

and the precedence order is

$$\rhd_h^Y = \begin{cases} \mathfrak{r} \rhd \mathfrak{c} & e \in \mathsf{d}(Y) \\ \mathfrak{c} \rhd \mathfrak{r} & \text{otherwise.} \end{cases}$$

The choice function $C^h$ induced by this tasks-and-slots preference structure does not satisfy the irrelevance of rejected contracts condition, as we have that $C^h(\{(d, h, c), (d, h, r)\}) = \{(d, h, c)\}$, while $C^h(\{(d, h, c), (d, h, r), (e, h, r)\}) = \{(d, h, r)\}$.

# References


Abdulkadiroğlu, A., P. A. Pathak, and A. E. Roth (2005). The New York City high school match. *American Economic Review 95*, 364–367. (Cited on page 5.)

Abdulkadiroğlu, A., P. A. Pathak, A. E. Roth, and T. Sönmez (2005). The Boston public school match. *American Economic Review 95*, 368–371. (Cited on page 5.)

Abdulkadiroğlu, A. and T. Sönmez (2003). School choice: A mechanism design approach. *American Economic Review 93*, 729–747. (Cited on page 2.)

Adachi, H. (2000). On a characterization of stable matchings. *Economics Letters 68*, 43–49. (Cited on page 14.)

Aizerman, M. A. and A. V. Malishevski (1981). General theory of best variants choice: Some aspects. *IEEE Transactions on Automatic Control 26*(5), 1030–1040. (Cited on page 26.)

Alkan, A. and D. Gale (2003). Stable schedule matching under revealed preference. *Journal of Economic Theory 112*, 289–306. (Cited on page 10.)

Ausubel, L. M. and P. R. Milgrom (2006). The lovely but lonely Vickrey auction. In P. Cramton, Y. Shoham, and R. Steinberg (Eds.), *Combinatorial Auctions*, Chapter 1, pp. 17–40. MIT Press. (Cited on page 2.)

Avery, C., C. Jolls, R. A. Posner, and A. E. Roth (2001). The market for federal judicial law clerks. *University of Chicago Law Review 68*, 793–902. (Cited on pages 2 and 10.)

Avery, C., C. Jolls, R. A. Posner, and A. E. Roth (2007). The new market for federal judicial law clerks. *University of Chicago Law Review 74*, 447–486. (Cited on pages 2 and 10.)

Aygün, O. and I. Bó (2017). College admission with multidimensional privileges: The Brazilian affirmative action case. WZB Berlin Working paper. (Cited on page 19.)

Aygün, O. and T. Sönmez (2012). Matching with contracts: The critical role of irrelevance of rejected contracts. Boston College working paper. (Cited on page 51.)

Aygün, O. and T. Sönmez (2013). Matching with contracts: Comment. *American Economic Review 103*(5), 2050–2051. (Cited on pages 22, 25, and 35.)

Aygün, O. and T. Sönmez (2014). The importance of irrelevance of rejected contracts in matching under weakened substitutes conditions. Boston College working paper. (Cited on page 25.)

Aygün, O. and B. Turhan (2016). Dynamic reserves in matching markets: Theory and applications. Boğaziçi University working paper. (Cited on pages 3, 19, and 24.)

Aygün, O. and B. Turhan (2017). Large-scale affirmative action in school choice: Admissions to IITs in India. *American Economic Review 107*(5), 210–13. (Cited on pages 3 and 19.)

Balinski, M. and T. Sönmez (1999). A tale of two mechanisms: Student placement. *Journal of Economic Theory 84*, 73–94. (Cited on page 2.)

Benoît, J.-P. and V. Krishna (2001). Multiple-object auctions with budget constrained bidders. *Review of Economic Studies 68*(1), 155–179. (Cited on pages 3 and 20.)

Bikhchandani, S. and J. Mamer (1997). Competitive equilibrium in an exchange economy with indivisibilities. *Journal of Economic Theory 74*, 385–413. (Cited on page 2.)

Blair, C. (1988). The lattice structure of the set of stable matchings with multiple partners. *Mathematics of Operations Research 13*, 619–628. (Cited on page 39.)

Chambers, C. P. and M. B. Yenmez (2017). Choice and matching. *American Economic Journal: Microeconomics 9*(3), 126–147. (Cited on page 26.)

Chambers, C. P. and M. B. Yenmez (2018). On lexicographic choice. *Economics Letters 171*, 222–224. (Cited on page 26.)

Che, Y.-K. and I. Gale (1998). Standard auctions with financially constrained bidders. *Review of Economic Studies 65*(1), 1–21. (Cited on pages 3 and 20.)

Dimakopoulos, P. D. and C.-P. Heller (2019). Matching with waiting times: The German entry-level labor market for lawyers. *Games and Economic Behavior 115*, 289–313. (Cited on pages 3 and 21.)

Dubins, L. E. and D. A. Freedman (1981). Machiavelli and the Gale-Shapley algorithm. *American Mathematical Monthly 88*, 485–494. (Cited on page 8.)

Echenique, F. (2012). Contracts vs. salaries in matching. *American Economic Review 102*, 594–601. (Cited on page 24.)

Fleiner, T. (2003). A fixed point approach to stable matchings and some applications. *Mathematics of Operations Research 28*, 103–126. (Cited on pages 14 and 23.)

Gale, D. and L. S. Shapley (1962). College admissions and the stability of marriage. *American Mathematical Monthly 69*, 9–15. (Cited on pages 8 and 16.)

Gul, F. and E. Stacchetti (1999). Walrasian equilibrium with gross substitutes. *Journal of Economic Theory 87*, 95–124. (Cited on page 2.)

Gul, F. and E. Stacchetti (2000). The English auction with differentiated commodities. *Journal of Economic Theory 92*, 66–95. (Cited on page 2.)

Hafalir, I. E., F. Kojima, and M. B. Yenmez (2019). Interdistrict school choice: A theory of student assignment. Stanford University working paper. (Cited on pages 20 and 21.)

Hassidim, A., A. Romm, and R. I. Shorrer (2016a). Redesigning the Israeli psychology masters match. Hebrew University working paper. (Cited on pages 3 and 18.)

Hassidim, A., A. Romm, and R. I. Shorrer (2016b). "Strategic" behavior in a strategy-proof environment. Hebrew University working paper. (Cited on pages 3 and 18.)

Hassidim, A., A. Romm, and R. I. Shorrer (2017). Redesigning the Israeli psychology master's match. *American Economic Review 107*(5), 205–09. (Cited on pages 3 and 18.)

Hassidim, A., A. Romm, and R. I. Shorrer (2018). Need vs. merit: The large core of college admissions markets. Hebrew University working paper. (Cited on pages 3, 20, and 24.)

Hatfield, J. W. and F. Kojima (2008). Matching with contracts: Comment. *American Economic Review 98*, 1189–1194. (Cited on pages 2, 3, and 14.)

Hatfield, J. W. and F. Kojima (2009). Group incentive compatibility for matching with contracts. *Games and Economic Behavior 67*, 745–749. (Cited on page 33.)

Hatfield, J. W. and F. Kojima (2010). Substitutes and stability for matching with contracts. *Journal of Economic Theory 145*, 1704–1723. (Cited on pages 3, 4, 14, 22, 23, and 42.)

Hatfield, J. W. and S. D. Kominers (2012). Matching in networks with bilateral contracts. *American Economic Journal: Microeconomics 4*, 176–208. (Cited on pages 2, 8, 13, 16, 29, 31, 32, and 33.)

Hatfield, J. W. and S. D. Kominers (2017). Contract design and stability in many-to-many matching. *Games and Economic Behavior 101*, 78–97. (Cited on pages 2, 8, 13, 16, 20, and 31.)

Hatfield, J. W. and S. D. Kominers (2018). Stability, strategy-proofness, and cumulative offer mechanisms. Harvard University working paper. (Cited on page 22.)

Hatfield, J. W., S. D. Kominers, A. Nichifor, M. Ostrovsky, and A. Westkamp (2013). Stability and competitive equilibrium in trading networks. *Journal of Political Economy 121*(5), 966–1005. (Cited on page 2.)

Hatfield, J. W., S. D. Kominers, and A. Westkamp (2019). Stability, strategy-proofness, and cumulative offer mechanisms. Harvard University working paper. (Cited on page 23.)

Hatfield, J. W. and P. Milgrom (2005). Matching with contracts. *American Economic Review 95*, 913–935. (Cited on pages 3, 7, 8, 10, 14, 17, 23, 26, 29, and 34.)

Jagadeesan, R. (forthcoming, 2019). Cadet-branch matching in a Kelso–Crawford economy. *American Economic Journal: Microeconomics*. (Cited on page 24.)

Kadam, S. V. (2017). Unilateral substitutability implies substitutable completability in many-to-one matching with contracts. *Games and Economic Behavior 102*, 56–68. (Cited on pages 4 and 22.)

Kagel, J. H. and A. E. Roth (2000). The dynamics of reorganization in matching markets: A laboratory experiment motivated by a natural experiment. *Quarterly Journal of Economics 115*, 201–235. (Cited on pages 2 and 10.)

Kamada, Y. and F. Kojima (2012). Stability and strategy-proofness for matching with constraints: A problem in the Japanese medical match and its solution. *American Economic Review 102*(3), 366–370. (Cited on page 21.)

Kamada, Y. and F. Kojima (2014). Efficient matching under distributional constraints: Theory and applications. *American Economic Review 105*(1), 67–99. (Cited on page 21.)

Kamada, Y. and F. Kojima (2018). Stability and strategy-proofness for matching with constraints: A necessary and sufficient condition. *Theoretical Economics 13*(2), 761–793. (Cited on page 21.)

Kelso, A. S. and V. P. Crawford (1982). Job matching, coalition formation, and gross substitutes. *Econometrica 50*, 1483–1504. (Cited on pages 7 and 15.)

Klaus, B. and M. Walzl (2009). Stable many-to-many matchings with contracts. *Journal of Mathematical Economics 45*(7-8), 422–434. (Cited on page 20.)

Klemperer, P. (2010). The product-mix auction: A new auction design for differentiated goods. *Journal of the European Economic Association 8*, 526–536. (Cited on page 2.)

Kojima, F., A. Tamura, and M. Yokoo (2018). Designing matching mechanisms under constraints: An approach from discrete convex analysis. *Journal of Economic Theory 176*, 803–833. (Cited on pages 21 and 24.)

Kominers, S. D. (2012). On the correspondence of contracts to salaries in (many-to-many) matching. *Games and Economic Behavior 75*, 984–989. (Cited on page 20.)

Kominers, S. D. and T. Sönmez (2015). Designing for diversity in matching. Harvard University working paper. (Cited on pages 3, 18, and 21.)

Kominers, S. D. and T. Sönmez (2016). Matching with slot-specific priorities: Theory. *Theoretical Economics 11*, 683–710. (Cited on pages 19 and 44.)

Kominers, S. D., A. Teytelboym, and V. P. Crawford (2017). An invitation to market design. *Oxford Review of Economic Policy 33*(4), 541–571. (Cited on page 10.)

Milgrom, P. (2004). *Putting Auction Theory to Work*. Cambridge University Press. (Cited on pages 2, 3, and 20.)

Milgrom, P. (2007). Package auctions and exchanges. *Econometrica 75*, 935–965. (Cited on page 2.)

Milgrom, P. R. (2009). Assignment messages and exchanges. *American Economics Journal: Microeconomics 1*(2), 95–113. (Cited on page 24.)

Milgrom, P. R. and B. Strulovici (2009). Substitute goods, auctions, and equilibrium. *Journal of Economic Theory 144*(1), 212–247. (Cited on page 24.)

Ostrovsky, M. (2008). Stability in supply chain networks. *American Economic Review 98*, 897–923. (Cited on pages 7 and 24.)

Ostrovsky, M. and R. Paes Leme (2015). Gross substitutes and endowed assignment valuations. *Theoretical Economics 10*(3), 853–865. (Cited on page 24.)

Pai, M. M. and R. Vohra (2014). Optimal auctions with financially constrained buyers. *Journal of Economic Theory 150*, 383–425. (Cited on pages 3 and 20.)

Pathak, P. A. (2017). What really matters in designing school choice mechanisms. In B. Honoré, A. Pakes, M. Piazzesi, and L. Samuelson (Eds.), *Advances in Economics and Econometrics, 11th World Congress of the Econometric Society*, pp. 176–214. (Cited on page 2.)

Pathak, P. A. and T. Sönmez (2008). Leveling the playing field: Sincere and sophisticated players in the Boston mechanism. *American Economic Review 98*, 1636–1652. (Cited on page 2.)

Roth, A. E. (1984). The evolution of the labor market for medical interns and residents: A case study in game theory. *Journal of Political Economy 92*, 991–1016. (Cited on pages 2 and 34.)

Roth, A. E. (1990). New physicians: A natural experiment in market organization. *Science 250*(4987), 1524–1528. (Cited on pages 2 and 10.)

Roth, A. E. (1991). A natural experiment in the organization of entry-level labor markets: Regional markets for new physicians and surgeons in the United Kingdom. *American Economic Review 81*, 415–440. (Cited on page 18.)

Roth, A. E. (2002). The economist as engineer: Game theory, experimentation, and computation as tools for design economics. *Econometrica 70*, 1341–1378. (Cited on pages 5 and 10.)

Roth, A. E. (2009). What have we learned from market design? *Innovation policy and the economy 9*(1), 79–112. (Cited on page 10.)

Roth, A. E. and X. Xing (1994). Jumping the gun: Imperfections and institutions related to the timing of market transactions. *American Economic Review 84*, 992–1044. (Cited on pages 2, 10, and 18.)

Roth, A. E. and X. Xing (1997). Turnaround time and bottlenecks in market clearing: Decentralized matching in the market for clinical psychologists. *Journal of Political Economy 105*, 284–329. (Cited on page 5.)

Sönmez, T. (2013). Bidding for army career specialties: Improving the ROTC branching mechanism. *Journal of Political Economy 121*, 186–219. (Cited on pages 3, 4, 21, and 22.)

Sönmez, T. and T. B. Switzer (2013). Matching with (branch-of-choice) contracts at United States Military Academy. *Econometrica 81*, 451–488. (Cited on pages 3, 4, 21, and 22.)

Sun, N. and Z. Yang (2006). Equilibria and indivisibilities: Gross substitutes and complements. *Econometrica 74*, 1385–1402. (Cited on page 2.)

Yenmez, M. B. (2018). A college admissions clearinghouse. *Journal of Economic Theory 176*, 859–885. (Cited on pages 3, 19, 20, and 24.)